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Parameterized Complexity of MinCSP

Marcin Pilipczuk
m.pilipczuk@mimuw.edu.pl

based on joint work with
Eun Jung Kim, Stefan Kratsch, and Magnus Wahlström
and many others

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CSP dichotomy

Theorem (Bulatov, Zhuk, 2017)

For every finite D and Γ, the corresponding CSP is either NP-complete and polynomial time solvable.

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$\textsc{MinCSP}(D, \Gamma)$: Can you delete k constraints to make the instance satisfiable?

- Interesting if $\text{CSP}(D, \Gamma)$ is P-time.
- Trivial $n^{\mathcal{O}(k)}$ algorithm.
- Is it FPT parameterized by k?

Min SAT

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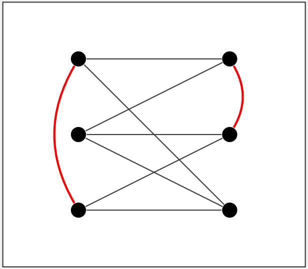

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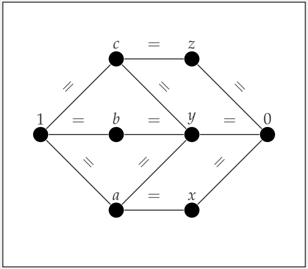

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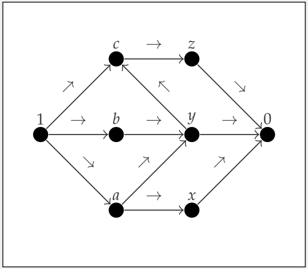

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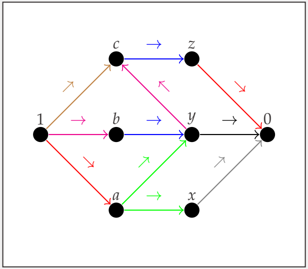

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    W[1]-hard (Marx, Razgon 2009)

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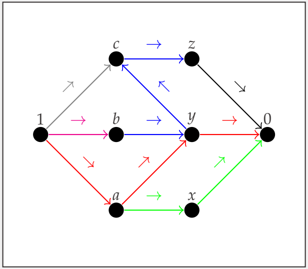

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- $D = \{0, 1\}$,
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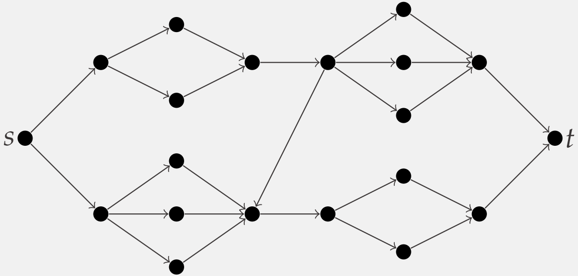
  - 3-Chain SAT!
    FPT status was open for some time.

Minimum (s,t)-cut



- Minimum (s,t)-cut problem: P-time.

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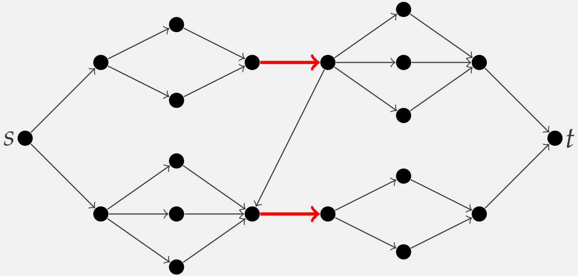

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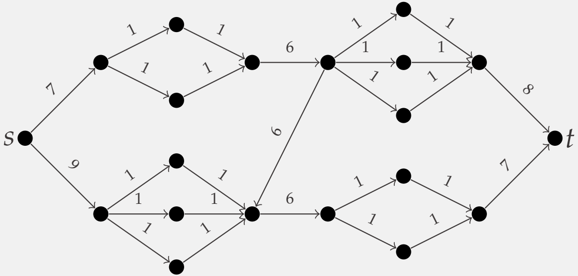

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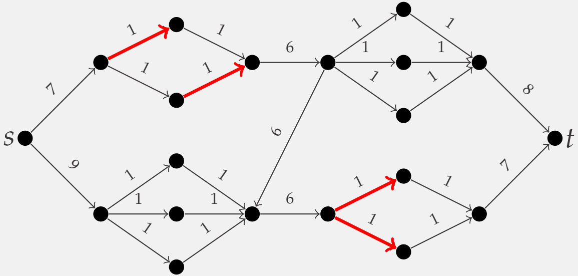

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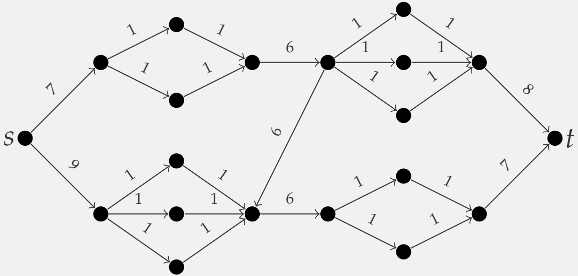

- Minimum (s, t)-cut problem: P-time.
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- What about bi-objective?
  - Cardinality $\leq k$. (Figure: $k = 3$.)
  - Minimum weight.

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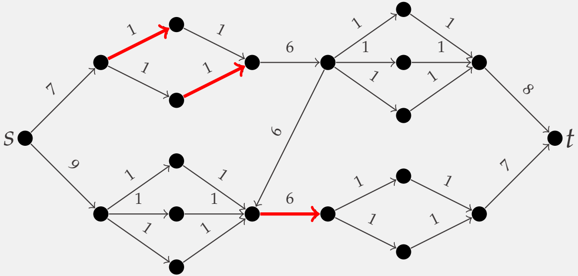

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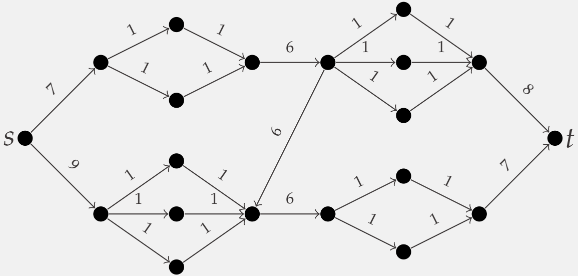

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- Still P-time if you add weights (capacities).
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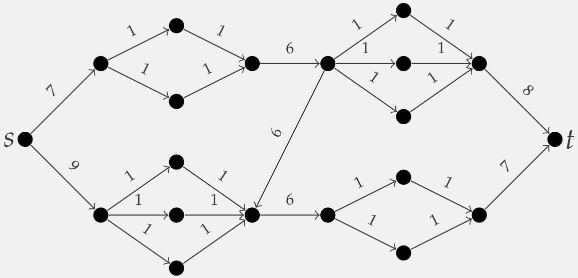
- This version is NP-hard.

Bi-objective (s, t)-cut



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Input: digraph $G, s, t \in V(G), \omega : E(G) \to \mathbb{Z}_+, k, W \in \mathbb{Z}$.
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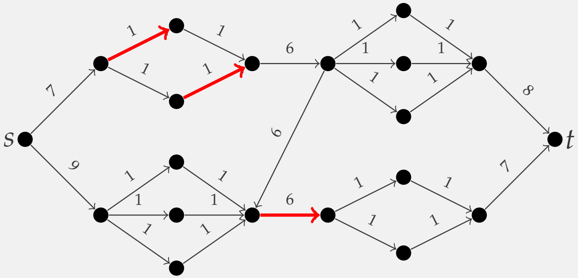

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- Figure: $k = 3, W = 8 \longrightarrow$ YES.

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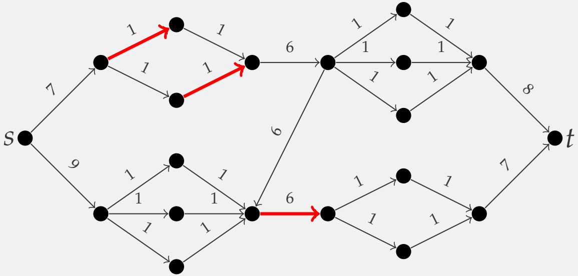

- NP-hard.
- FPT when parameterized by $k + W$.
    - Multi-budgeted important separators.
    - Kratsch, Li, Marx, P., Wahlström, IPEC 2018.

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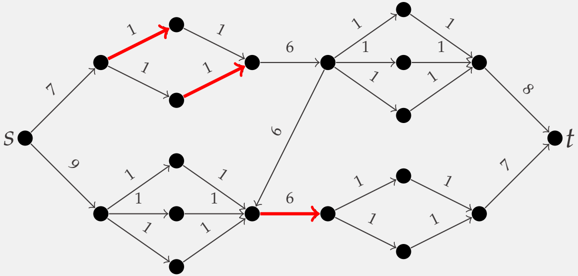

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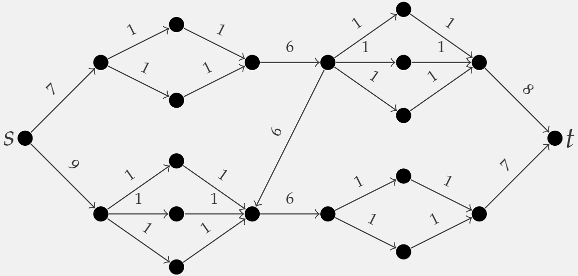
- Parameterization by k only?
  - Undirected: can hammer down with randomized contractions / treewidth reductions.



- Important observation: P-time for $k = \lambda_G(s,t)$.
  - Example: $k = 2$.

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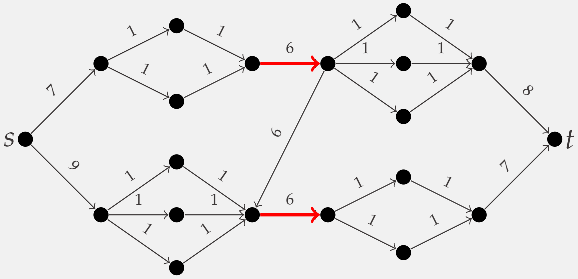

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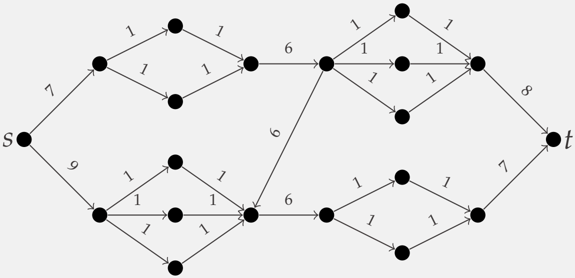

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- Set

$$M := 1 + \sum_{e \in E(G)} \omega(e),$$

$$\mathrm{cap}(e) := M + \omega(e),$$

and ask for cut of capacity $\leq kM + W$.

Directed flow-augmentation

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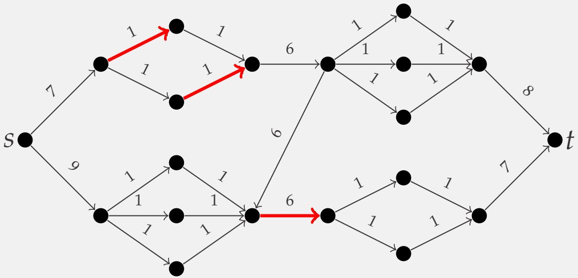

Theorem (Kim, Kratsch, P., Wahlström)

There exists a randomized polynomial-time algorithm that, given a digraph G, $s, t \in V(G)$, and $k \in \mathbb{Z}$, outputs $A \subseteq V(G) \times V(G)$ so that for every minimal (s,t)-cut Z of $|Z| \leq k$, with probability $2^{-\mathcal{O}(k^4 \log k)}$ the set Z is a minimum (s,t)-cut in $G + A$.

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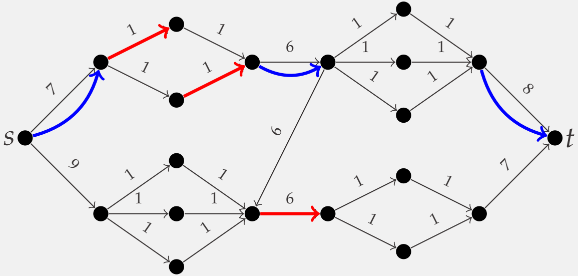

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- Repeat $2^{\mathcal{O}(k^4 \log k)}$ times:
  - Invoke flow-augmentation, obtaining A.
  - Find min-weight solution of cardinality $\lambda_{G+A}(s,t)$ in $G + A$.
- Bi-objective (s,t)-cut is randomized FPT when parameterized by k.

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- Works in slightly more general setting of *star* (s, t)*-cut*.
  - (s, t)-cut whose every arc goes from reachable-from-s to non-reachable-from-s.
- Undirected graphs: $2^{-\mathcal{O}(k \log k)}$ success probability.

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  Path $P_i \sim$ variable x_i where it is cut.

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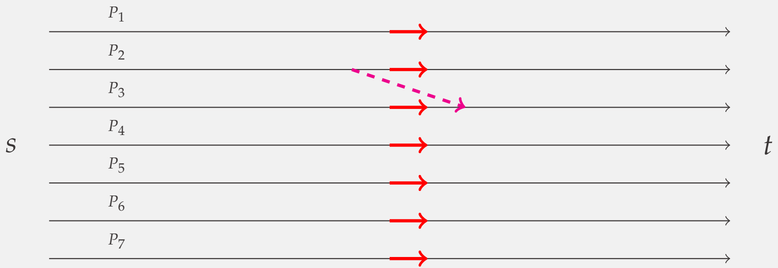

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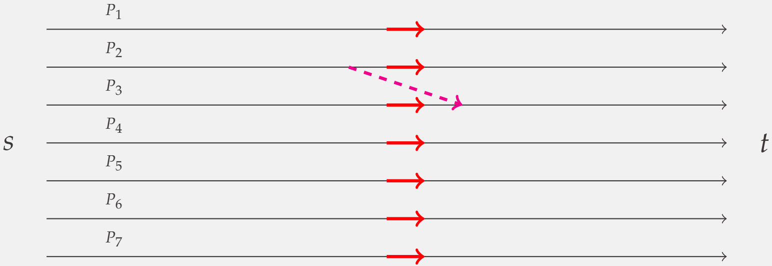

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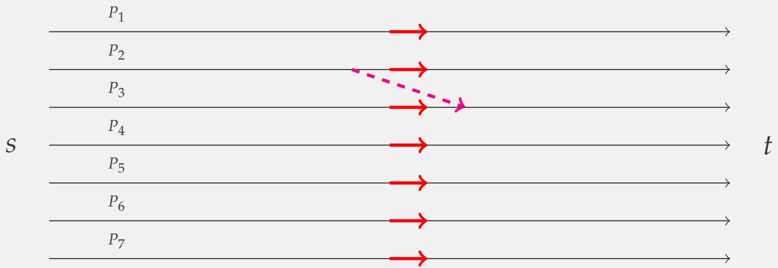

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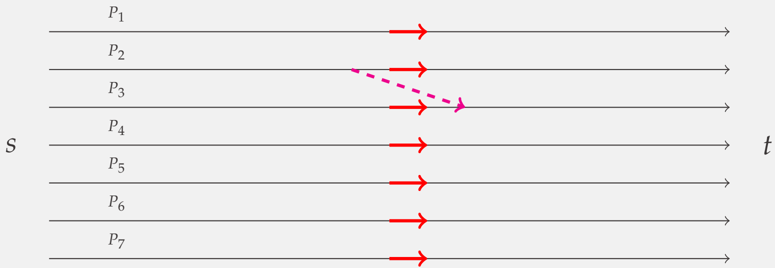

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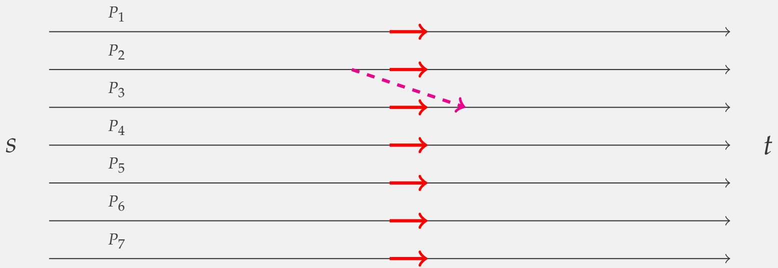

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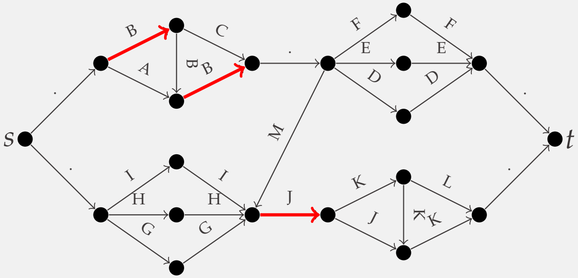
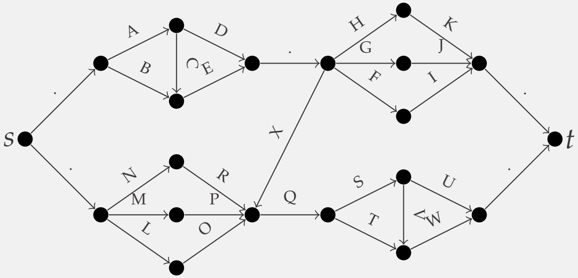
- Write such clause for every choice of (i, j, u, v) for $\exists Q$.
- Space of all CSP solutions $\equiv$ space of all st-mincuts.
- Flow-augmentation: cover the space of *minimal st*-cuts of size $\leq k$ with small number of such CSP instances.



Bundled cut

Input: digraph G, $s, t \in V(G)$, $k \in \mathbb{Z}$, family $\mathcal{B}$ of pairwise disjoint subsets of $E(G)$.
Question: does there exist a minimal (s,t)-cut $Z \subseteq \bigcup \mathcal{B}$ with $|\{B \in \mathcal{B} \mid B \cap Z \neq \emptyset\}| \leq k$.

Figure: solution of cost 2.



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All $B \in \mathcal{B}$ singletons $\longrightarrow$ minimum (s,t)-cut.

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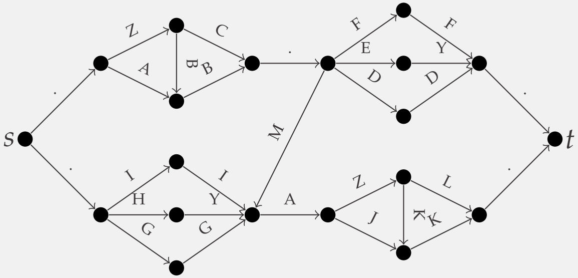

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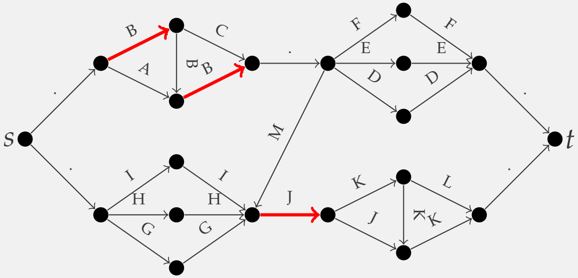
Every $B \in \mathcal{B}$ of size at most 2 $\longrightarrow$ $W[1]$-hard with param. k.

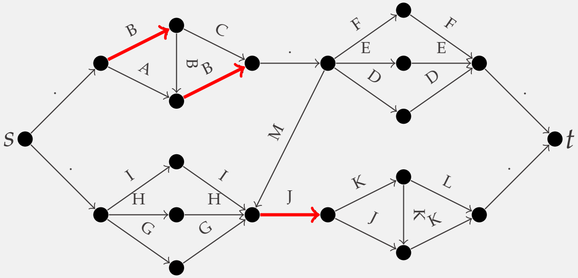
Bundled Cut



Bundled cut

Input: digraph $G, s, t \in V(G), k \in \mathbb{Z}$, family $\mathcal{B}$ of pairwise disjoint subsets of $E(G)$.
Question: does there exist a minimal (s,t)-cut $Z \subseteq \bigcup \mathcal{B}$ with $|\{B \in \mathcal{B} \mid B \cap Z \neq \emptyset\}| \leq k$.

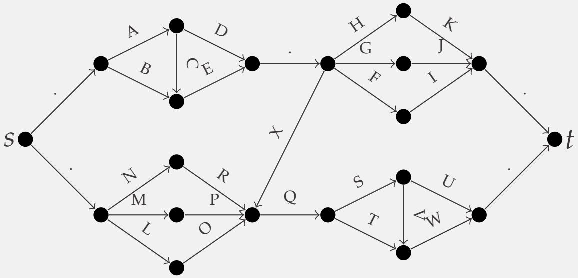
Every $B \in \mathcal{B}$ is a path of length $\leq \ell \longrightarrow \ell$-CHAIN SAT.



ℓ-Chain SAT

Input: digraph $G, s, t \in V(G), k \in \mathbb{Z}$, family $\mathcal{B}$ of pairwise disjoint subsets of $E(G)$, each being a path of length $\leq \ell$.
Question: does there exist a minimal (s,t)-cut $Z \subseteq \bigcup \mathcal{B}$ with $|\{B \in \mathcal{B} \mid B \cap Z \neq \emptyset\}| \leq k$.

Figure: $\ell = 3$.



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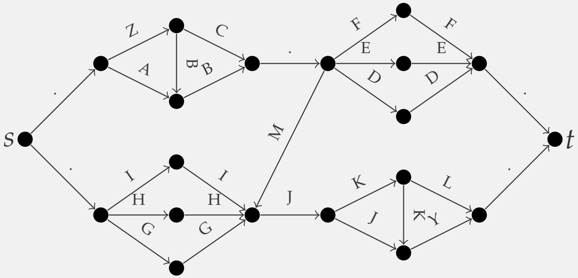
Case $\ell = 1$: Minimum (s,t)-Cut



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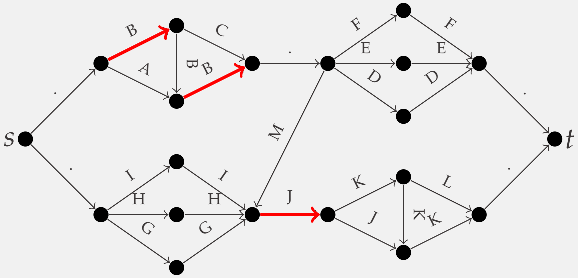
Question: does there exist a minimal (s,t)-cut $Z \subseteq \bigcup \mathcal{B}$ with $|\{B \in \mathcal{B} \mid B \cap Z \neq \emptyset\}| \leq k$.

Case $\ell = 2$: still Minimum (s,t)-Cut.



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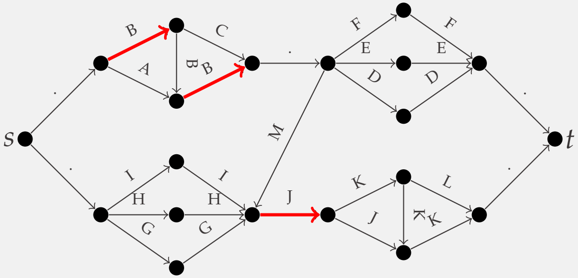
Case $\ell = 3$: starts to be interesting!

Chain SAT

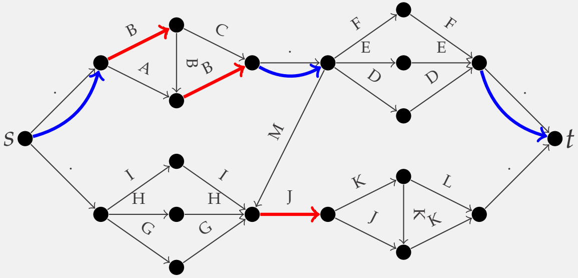


3-Chain SAT

Input: digraph G, $s, t \in V(G)$, $k \in \mathbb{Z}$, family $\mathcal{B}$ of pairwise disjoint subsets of $E(G)$, each being a path of length ≤ 3.
Question: does there exist a minimal (s,t)-cut $Z \subseteq \bigcup \mathcal{B}$ with $|\{B \in \mathcal{B} \mid B \cap Z \neq \emptyset\}| \leq k$.

Focus on case $\ell = 3$.

Chain SAT



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Flow-augmentation: focus on the case Z is a mincut ($|Z| \leq 2k$).

3-Chain SAT

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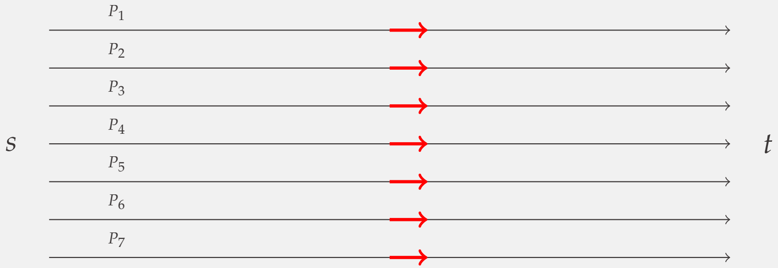

3-Chain SAT

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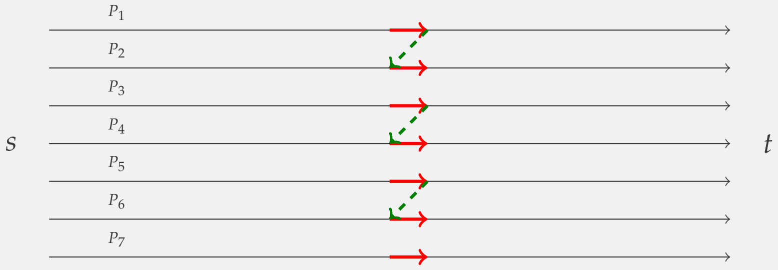
- Guess how cuts on paths group in $\mathcal{B}$.



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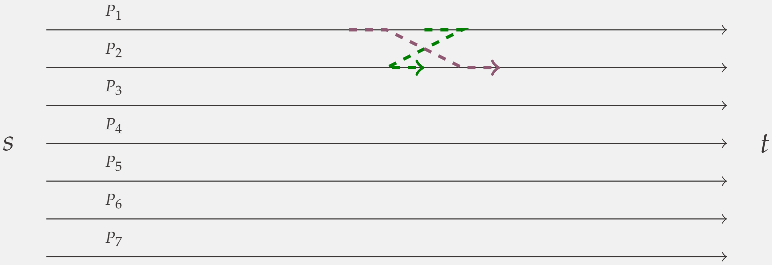

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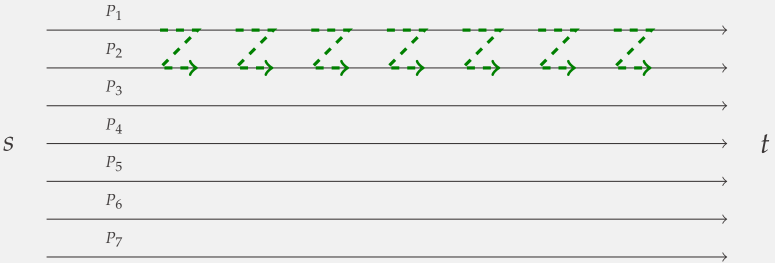

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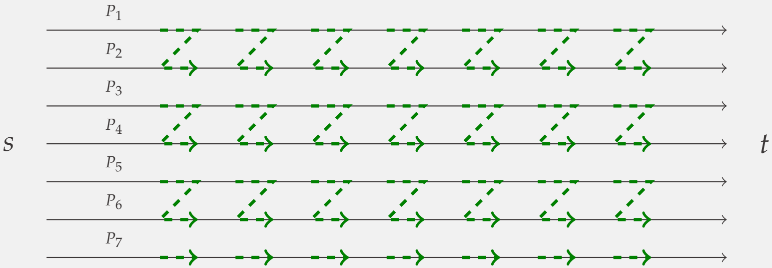

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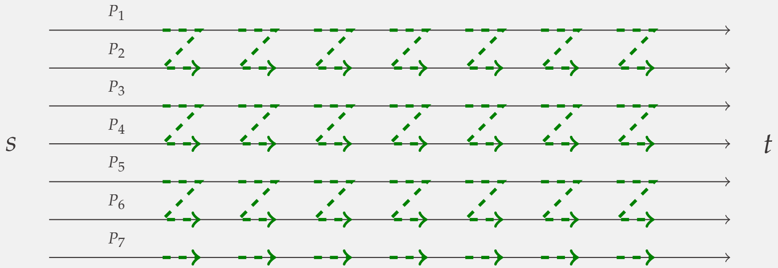

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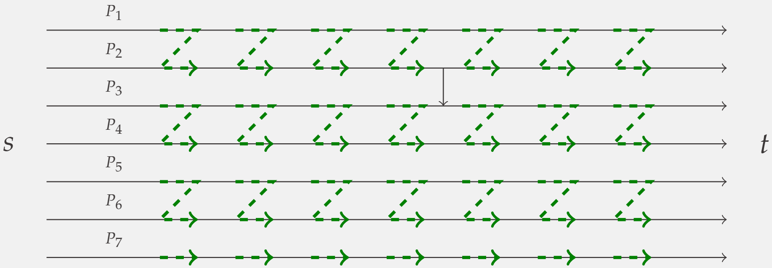
UNIVERSITY OF WARSAW



- Guess how cuts on paths group in $\mathcal{B}$.
- On one pair, candidates cannot cross.
- Collapse pair into one flow path.
- The same (P-time) description as the space of all mincuts.

ℓ-Chain SAT

Theorem (Kim, Kratsch, P., Wahlström)

CHAIN ℓ-SAT *is FPT when param. by k and* ℓ.

Works also in the weighted setting.
(Every $B \in \mathcal{B}$ has its weight and we do not want to exceed total weight budget.)

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Theorem (Kim, Kratsch, P., Wahlström)

(WEIGHTED) BUNDLED CUT WITH PAIRWISE LINKED DELETABLE EDGES *is FPT when param. by k and maximum bundle size.*

For every $B \in \mathcal{B}$ and $e, f \in B$, there exists a directed path between an endpoint of e and an endpoint of f (in one of the directions) that does not use edge of a different bundle.

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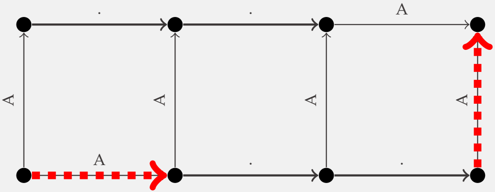

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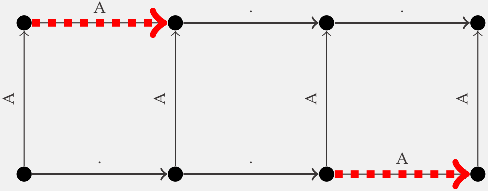

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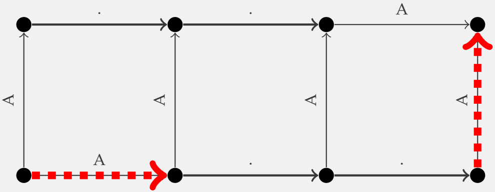

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Min SAT dichotomy

UNIVERSITY OF WARSAW

Theorem (Kim, Kratsch, P., Wahlström, SODA 2023)

FPT/W[1]-hard dichotomy for MIN SAT *for boolean domain.*

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- ℓ-CHAIN SAT was a roadblock.
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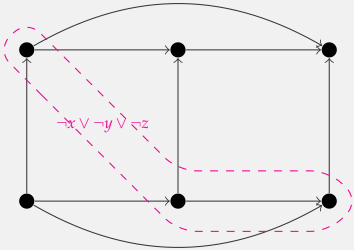
- Hardness: Marx-Razgon reduction is **the** hardness reduction.

Tractability Island 1



- Constraints defined as an AND of:
  - implications (can use constants 1 and 0);
  - ORs of negated variables.

  such that the implication graph is $2K_2$-free.

Then, MIN SAT is FPT when parameterized by k and max constraint arity (but only unweighted here; weighted is W[1]-hard).

Tractability Island 2

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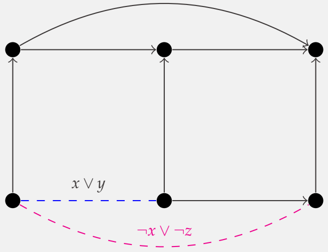

- Constraints defined as an AND of 2-clauses (can use constants 1 and 0).
- Assumption: for every constraint, the graph of 2-clauses is $2K_2$-free (after deleting 1 and 0).

Then, Min UnSAT is FPT when parameterized by k and max constraint arity (also in the weighted setting).

MinCSP
parameterized complexity
classification programme

Goal: Provide FPT vs W[1]-hard dichotomy theorems for MinCSP parameterized by the number of unsatisfied constraints for various classes of languages Γ.

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Note: requires $\text{CSP}(\Gamma)$ to be P-time.

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- Weighted Directed Feedback Arc Set
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Weighted Skew Multicut

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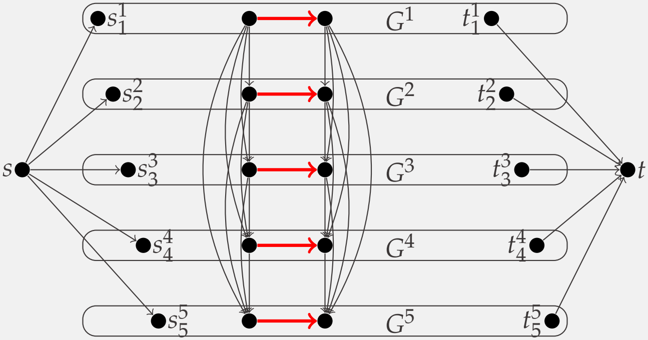

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(Weighted) Directed Subset Feedback Arc Set *reduces to* (Weighted) BCwPLDE.

Directed Subset Feedback Arc Set: the input digraph G is equipped with $R \subseteq E(G)$ and the goal is to delete at most k arcs so that no cycle contains an arc of R.

Undirected Multicut

- UNDIRECTED MULTICUT: Graph G, terminal pairs $\mathcal{T} \subseteq \binom{V(G)}{2}$, $k \in \mathbb{Z}$; delete at most k arcs so that every $st \in \mathcal{T}$ is separated.

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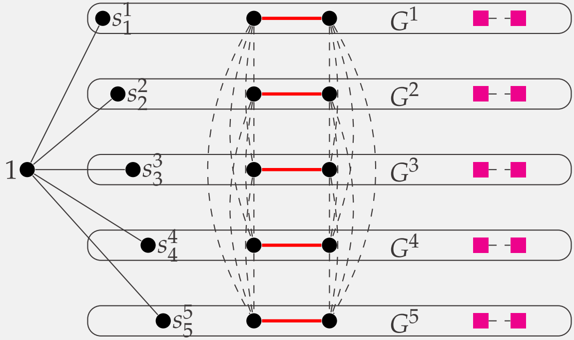

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Theorem (Kim, Masařík, P., Sharma, Wahlström)

WEIGHTED UNDIRECTED MULTICUT *can be solved using the algorithm of Tractability Island 2 as a black-box.*

Beyond boolean domain

- Point Line Algebra:
  - Domain: $\mathbb{Q}$.
  - Constraints have access to $<, \leq, =, \neq$ (and are FO formulae).

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- (Osipov, P., Wahlström, ESA 2024)
  $\Gamma = \{<, \neq, =\}$ is FPT.
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  - Graph formulation: Symmetric Multicut. Directed graph G, unordered pairs of terminals $\mathcal{T}$, integer k. Delete k edges so that for every $st \in \mathcal{T}$, s and t are not in the same strong component.

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- Most cases beyond look W[1]-hard.

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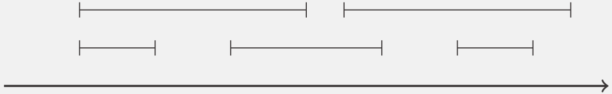
  - Domain: $\mathbb{Q}$.
  - Constraints have access to $<, \leq, =, \neq$ (and are FO formulae).
- $\Gamma = \{<, \leq\}$ is equivalent to DIRECTED SUBSET FEEDBACK ARC SET!
- (Osipov, P., Wahlström, ESA 2024)
  $\Gamma = \{<, \neq, =\}$ is FPT.
  $\Gamma = \{<, \leq, \neq, =\}$ is W[1]-hard.
- (Osipov, Wahlström, ESA 2023) Full dichotomy for FO over $\{=, \neq\}$.
- Most cases beyond look W[1]-hard.
- **OPEN**: Full dichotomy.

Beyond boolean domain



- Allen Algebra:
  - Domain: intervals $\{[a, b] \mid a, b \in \mathbb{Q}, a < b\}$.
  - Basic constraints: “precedes”, ”starts”, ”ends”, ”meets”, ”during”, ”overlap”, ”equal”.

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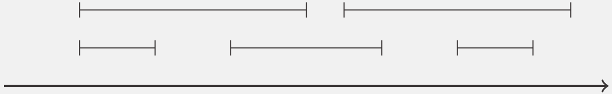

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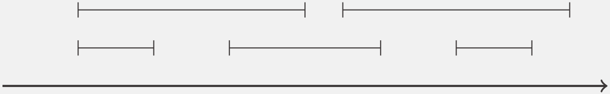

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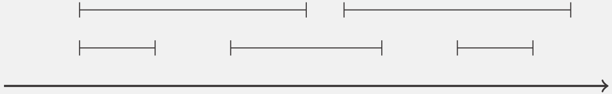

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    - Highlight 2: {“precedes”, “starts”, “equals”} has an FPT algorithm via reduction to BCwPLDE.

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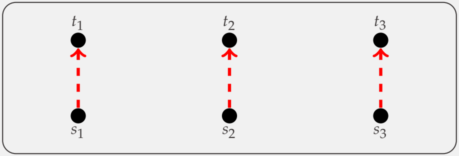
- (Dabrowski, Jonsson, Ordyniak, Osipov, Wahlström, SODA 2023)
  - FPT if two variables per equation and $\mathbb{K}$ is an Euclidean domain;
  - W[1]-hard for three variables per equation;
  - W[1]-hard for some commutative rings (e.g., $\mathbb{Z}/6\mathbb{Z}$).

3-terminal Directed Multicut



MULTICUT: graph G, terminal pairs $(s_i, t_i)_{i=1}^{\ell}$, integer k.
Goal: delete k edges so that no $s_i \to t_i$ path remains.

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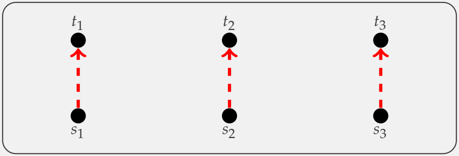

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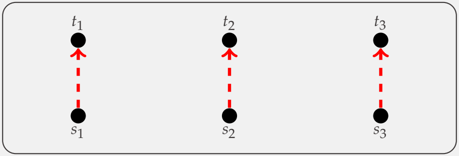

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Theorem (Hatzel, Jaffke, Lima, Masařík, P., Sharma, Sorge, SODA'23)

$\ell = 3$ case is FPT! (Uses twin-width and flow augmentation.)

Summary

Flow augmentation looks like the missing tool in directed graph separation problems.

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Closed some dichotomies and long-standing open problems.

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Thanks!