from the other side

PTAS for CSPs

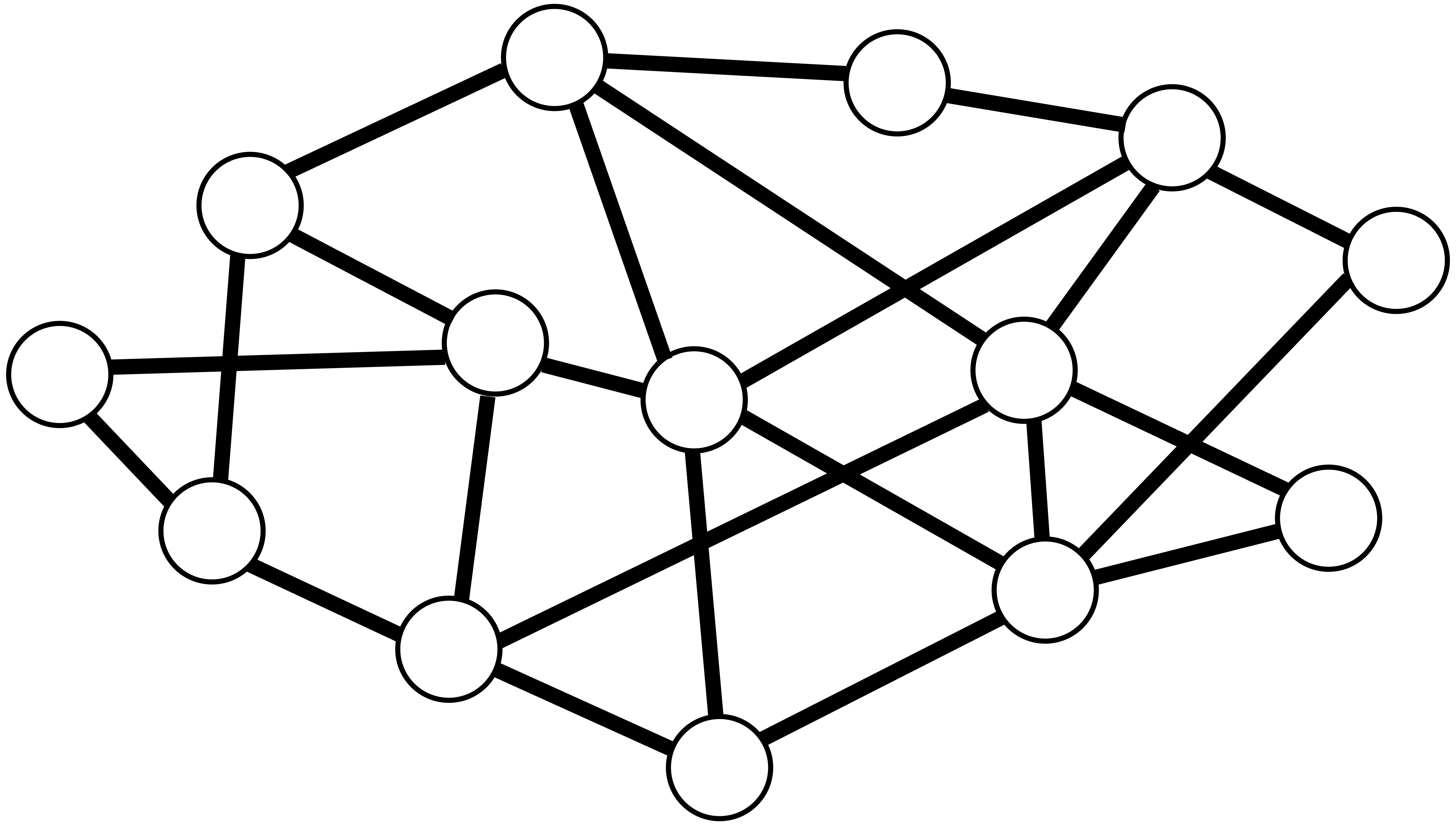
Standa Živný

PACS, Tallinn

7th July 2024

2-Colour

2-Colour



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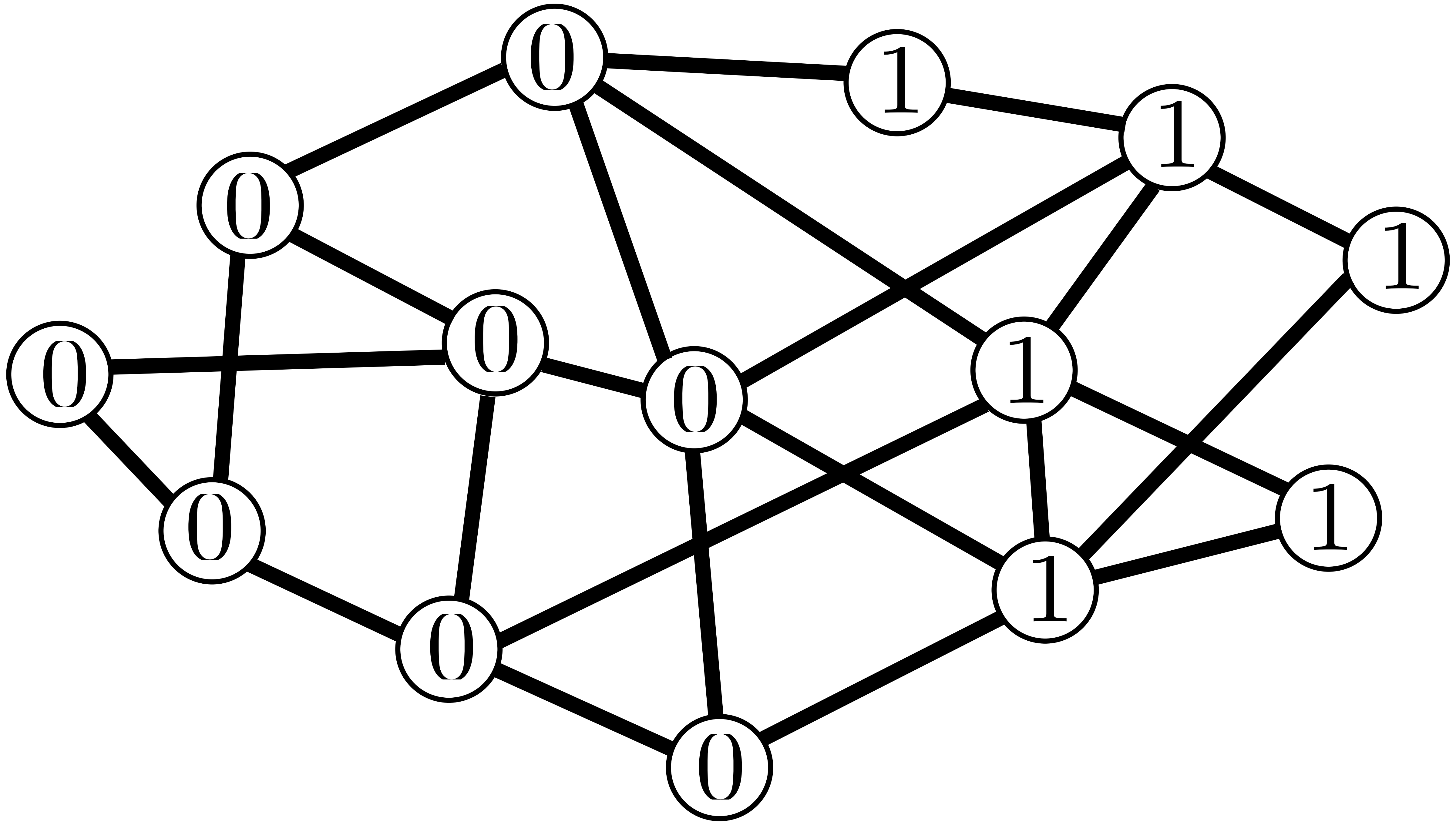
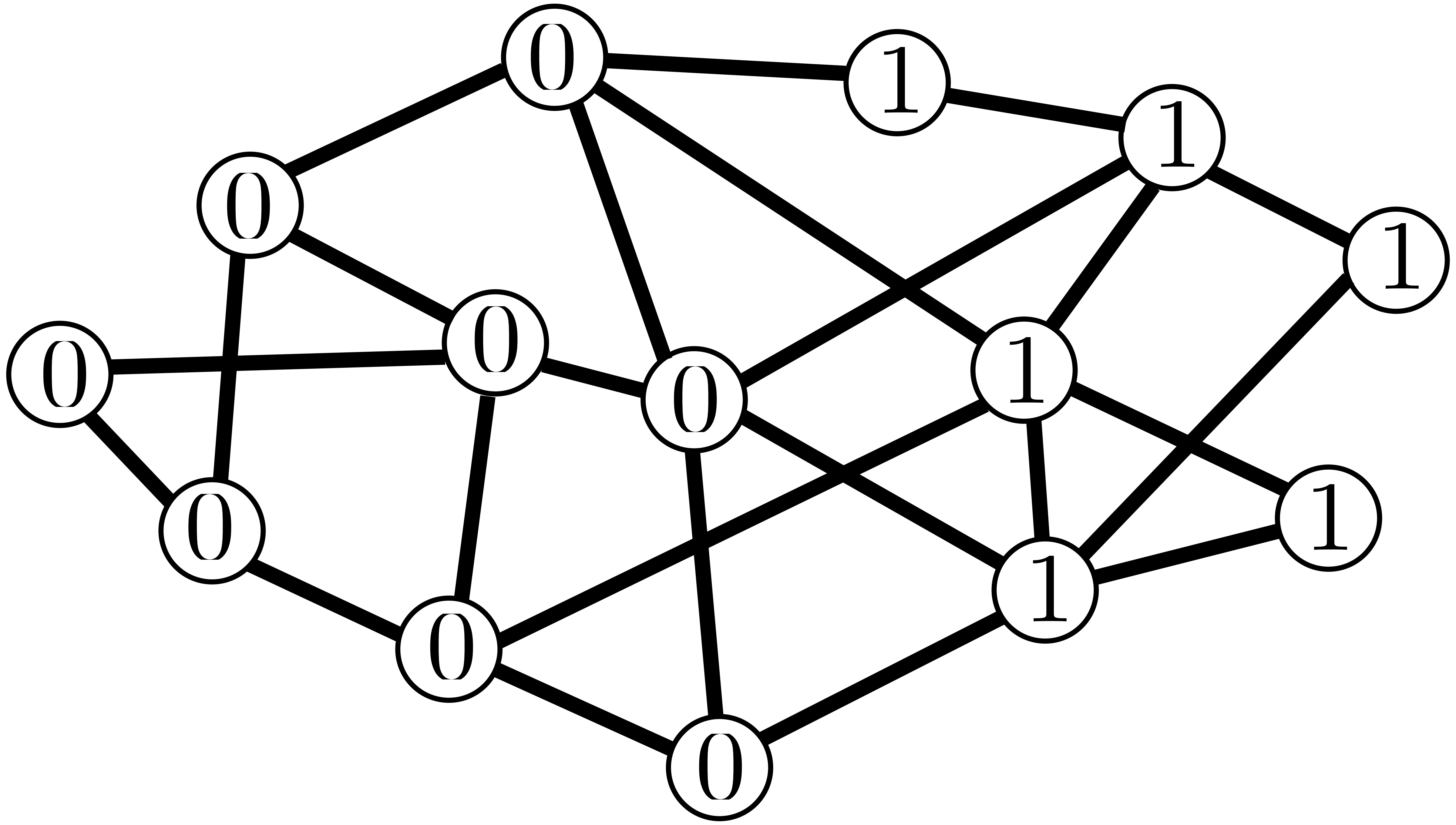
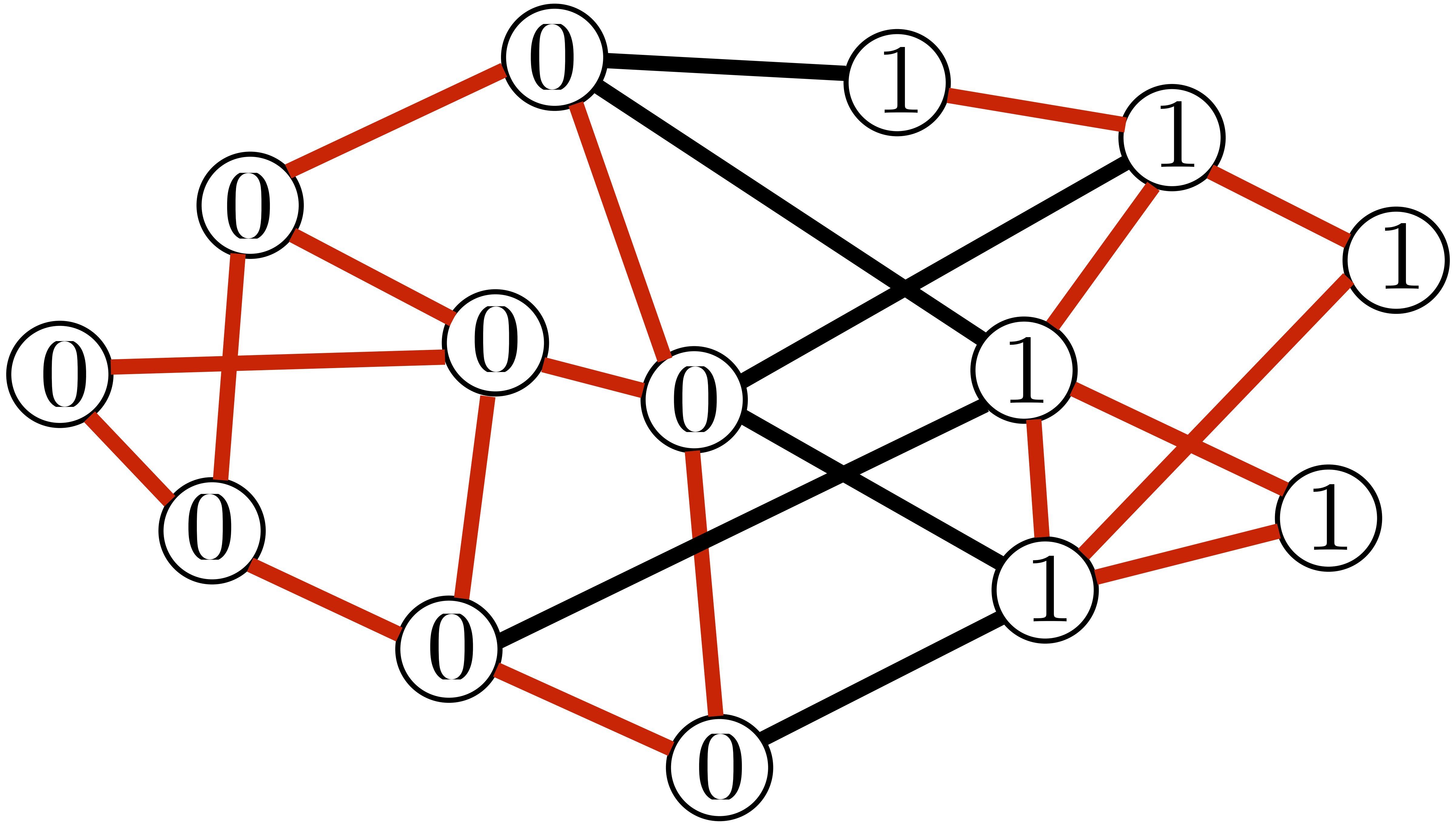


2-Colour

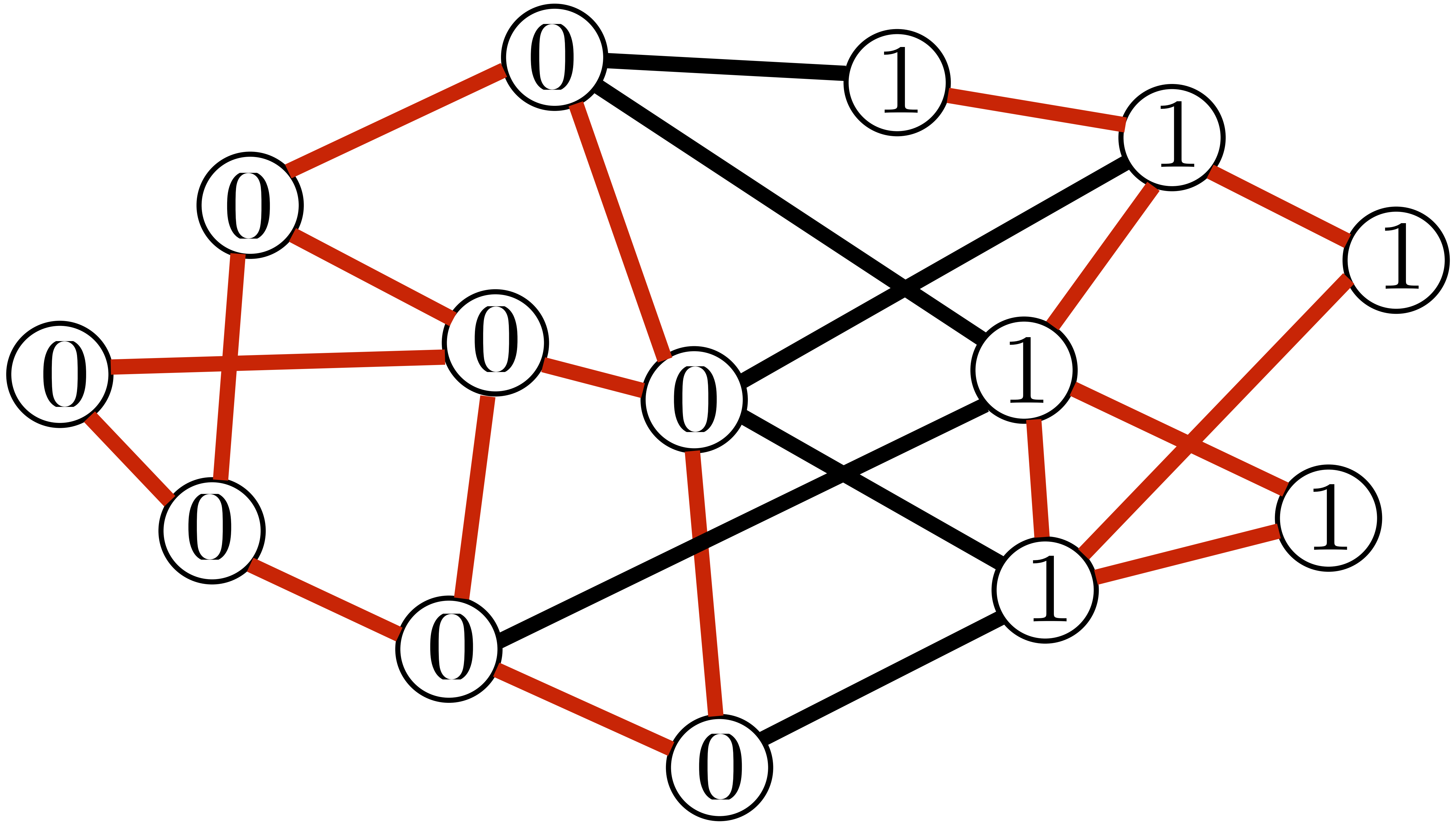


- PTIME

Min-UnCut



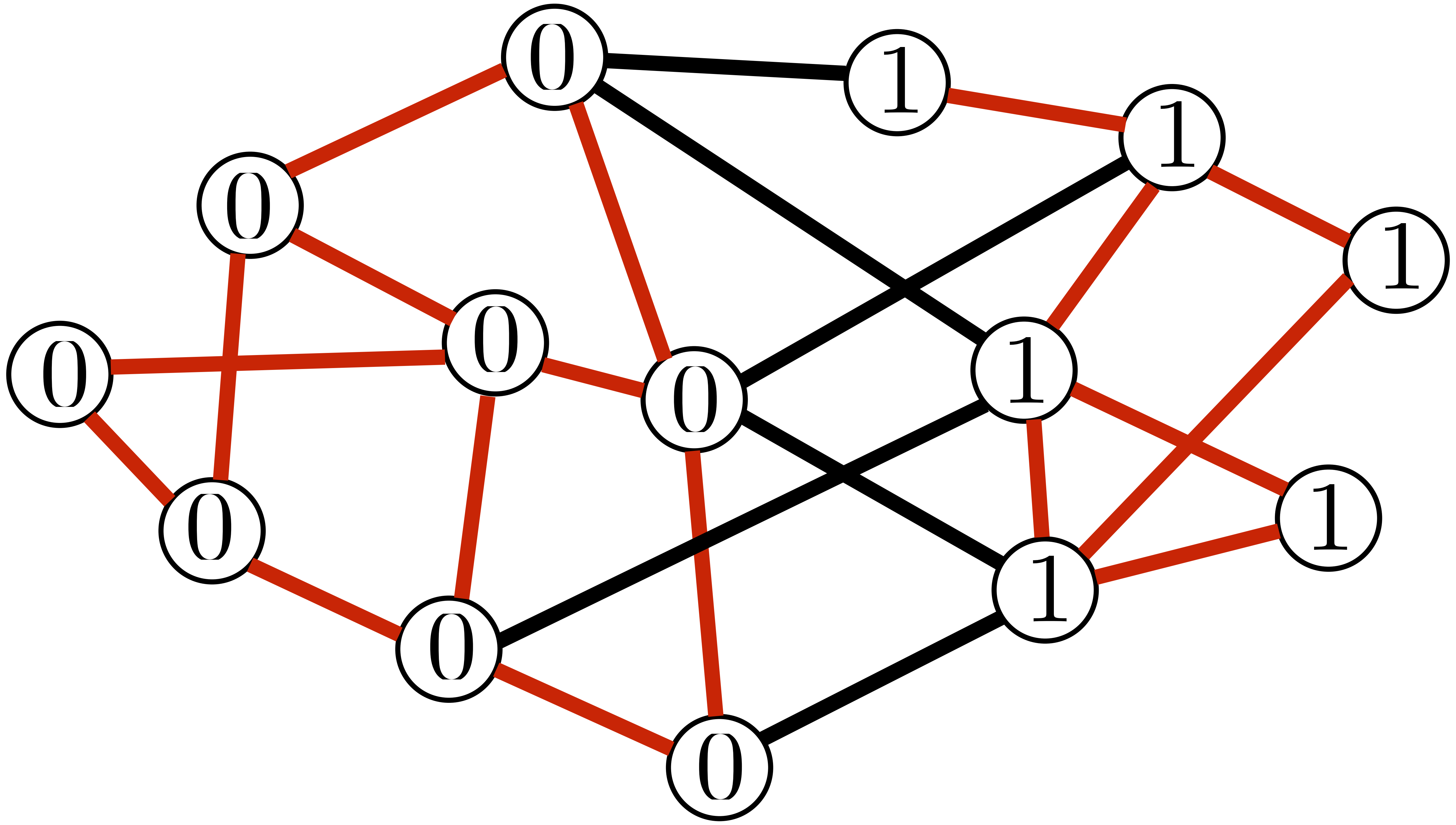
Min-UnCut



- APX-hard

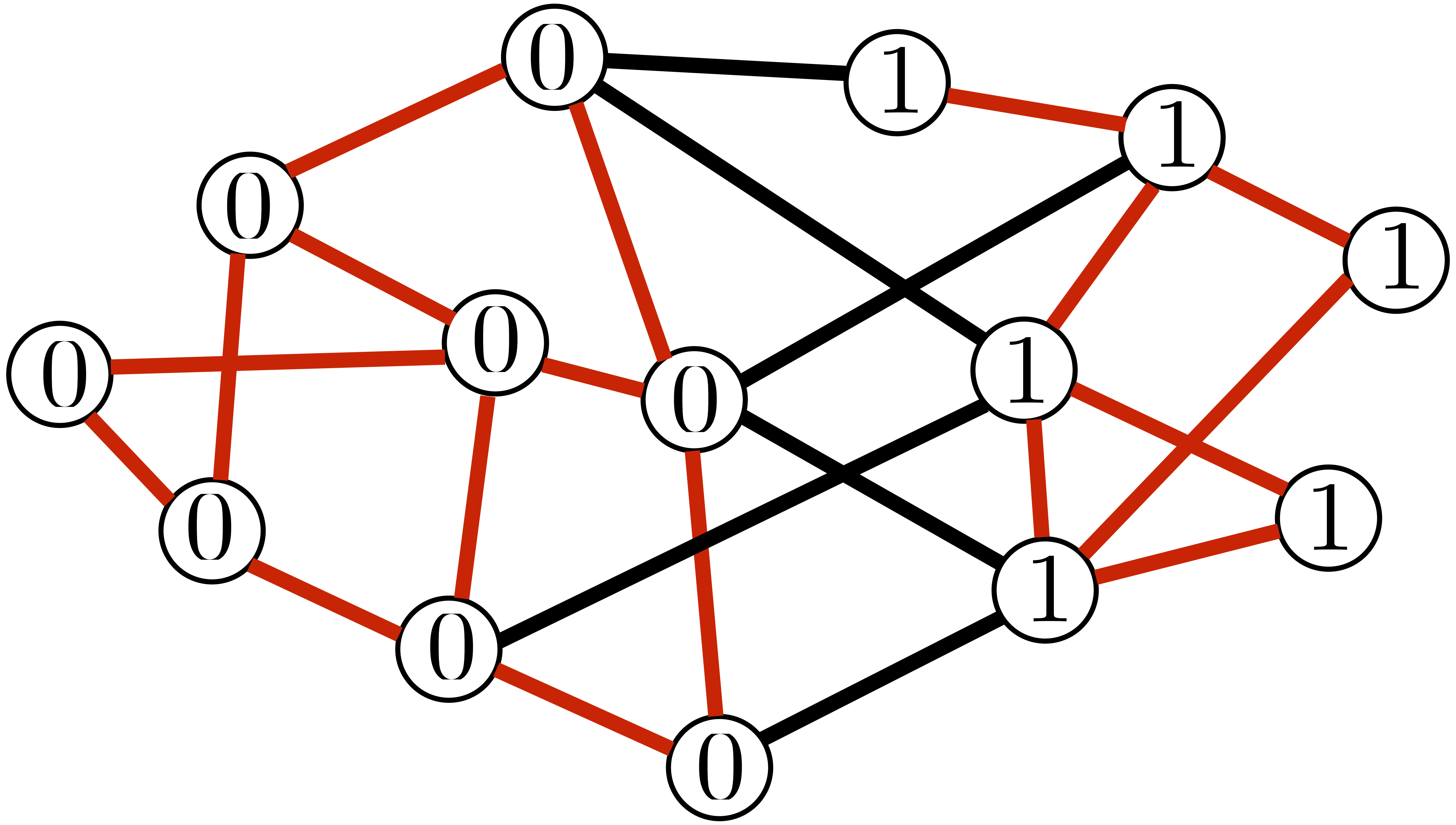
[Papadimitriou-Yannakakis JCSS'01]

Min-UnCut



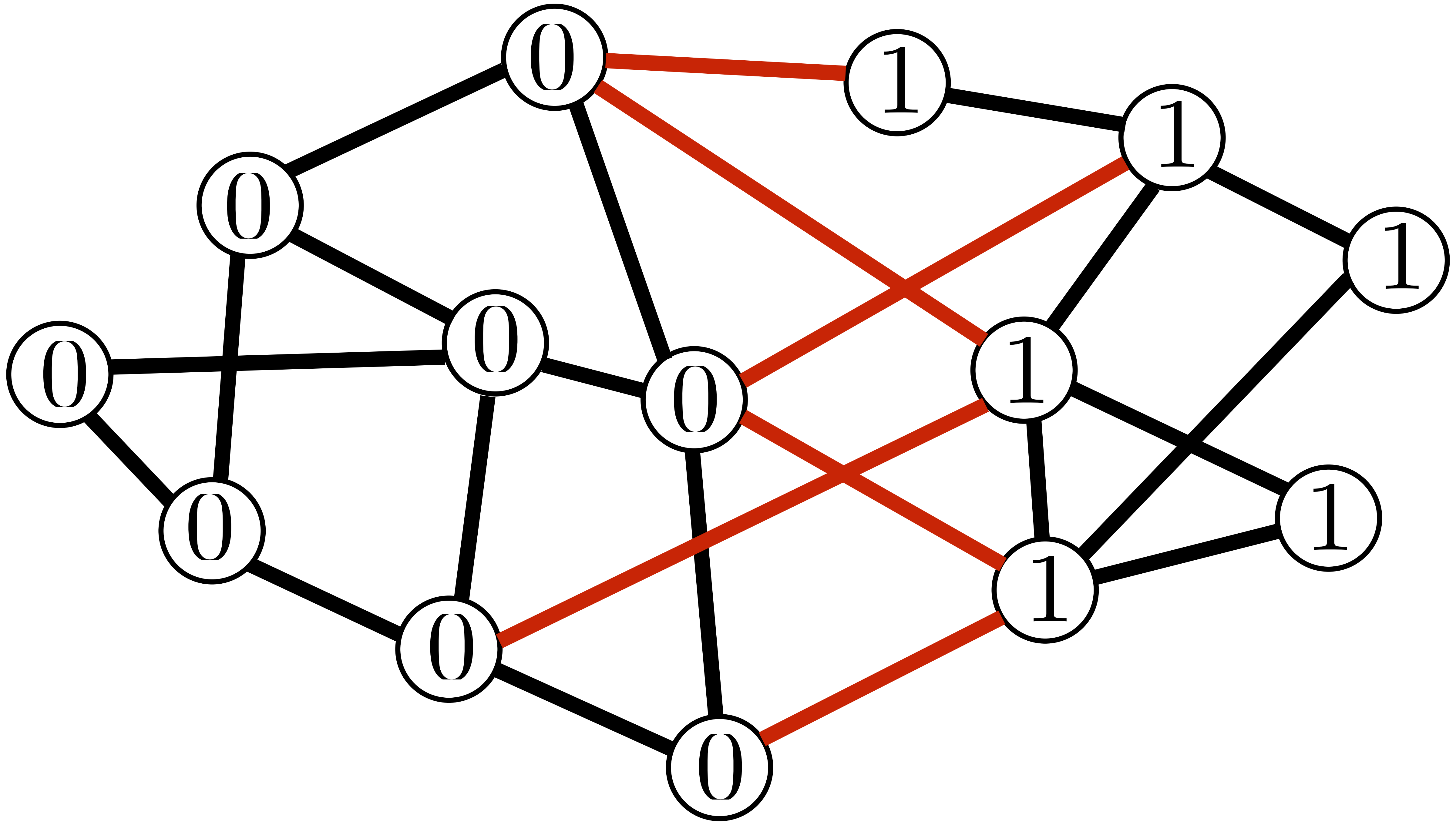
- APX-hard — *[Papadimitriou-Yannakakis JCSS'01]*
- $O(\sqrt{\log n})$-approx — *[Agarwal et al. STOC'05]*

Min-UnCut

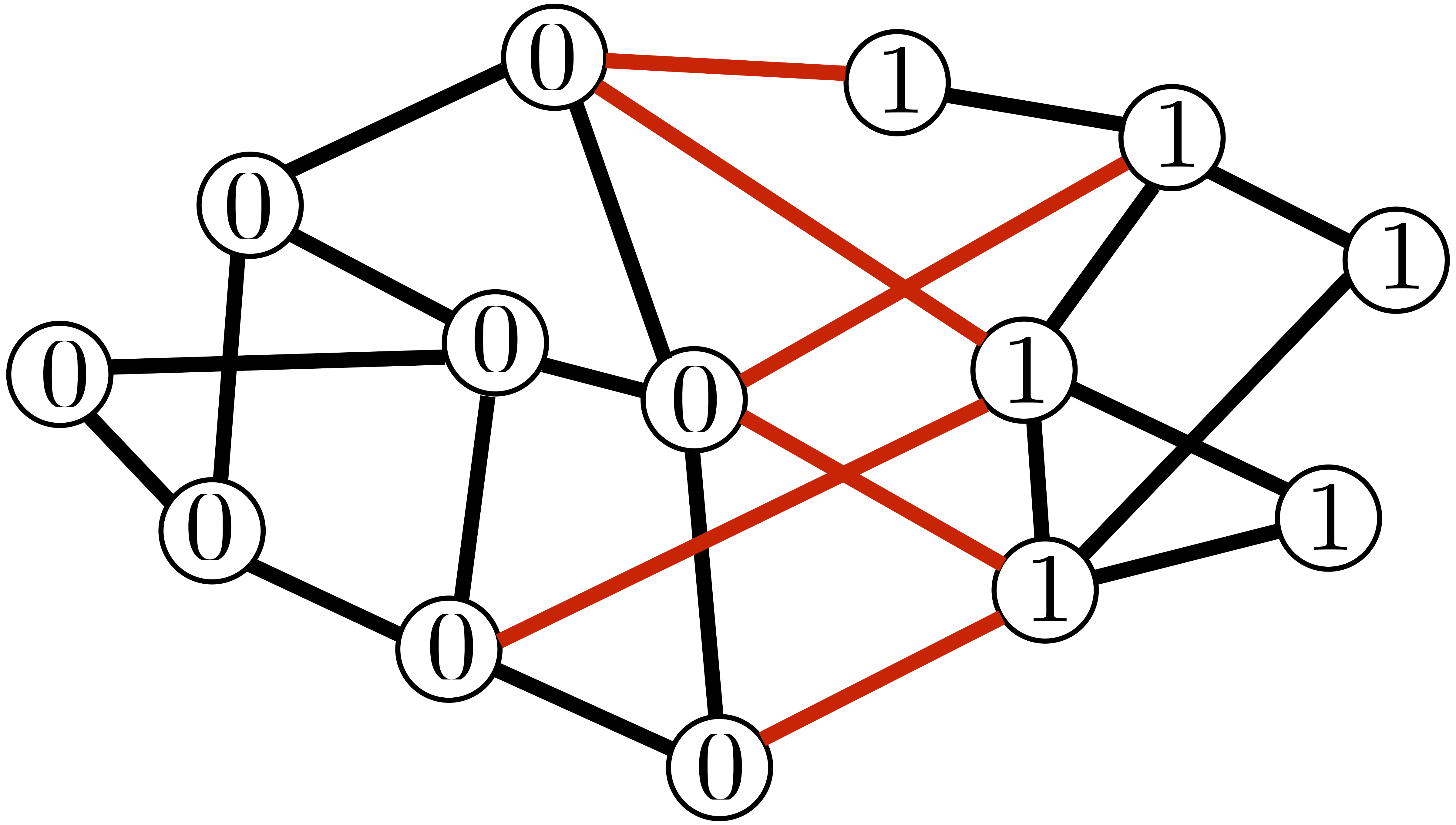


- APX-hard — *[Papadimitriou-Yannakakis JCSS'01]*
- $O(\sqrt{\log n})$-approx — *[Agarwal et al. STOC'05]*
- no O(1)-approx, under UGC — *[Chawla et al. CC'06, Khot-Vishnoi JACM'05]*

Max-Cut



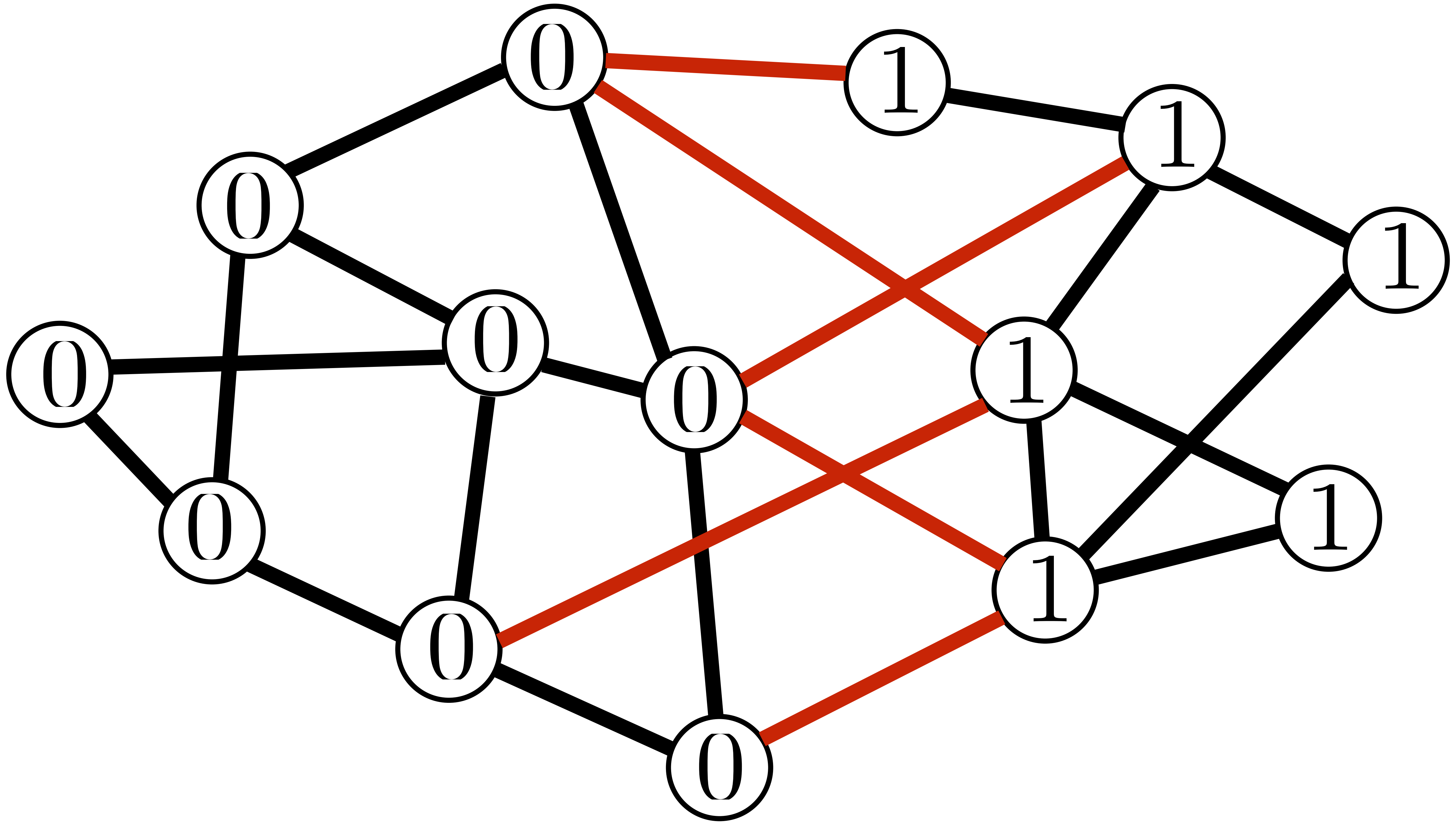
Max-Cut



- APX-complete

[Papadimitriou-Yannakakis JCSS'01]

Max-Cut

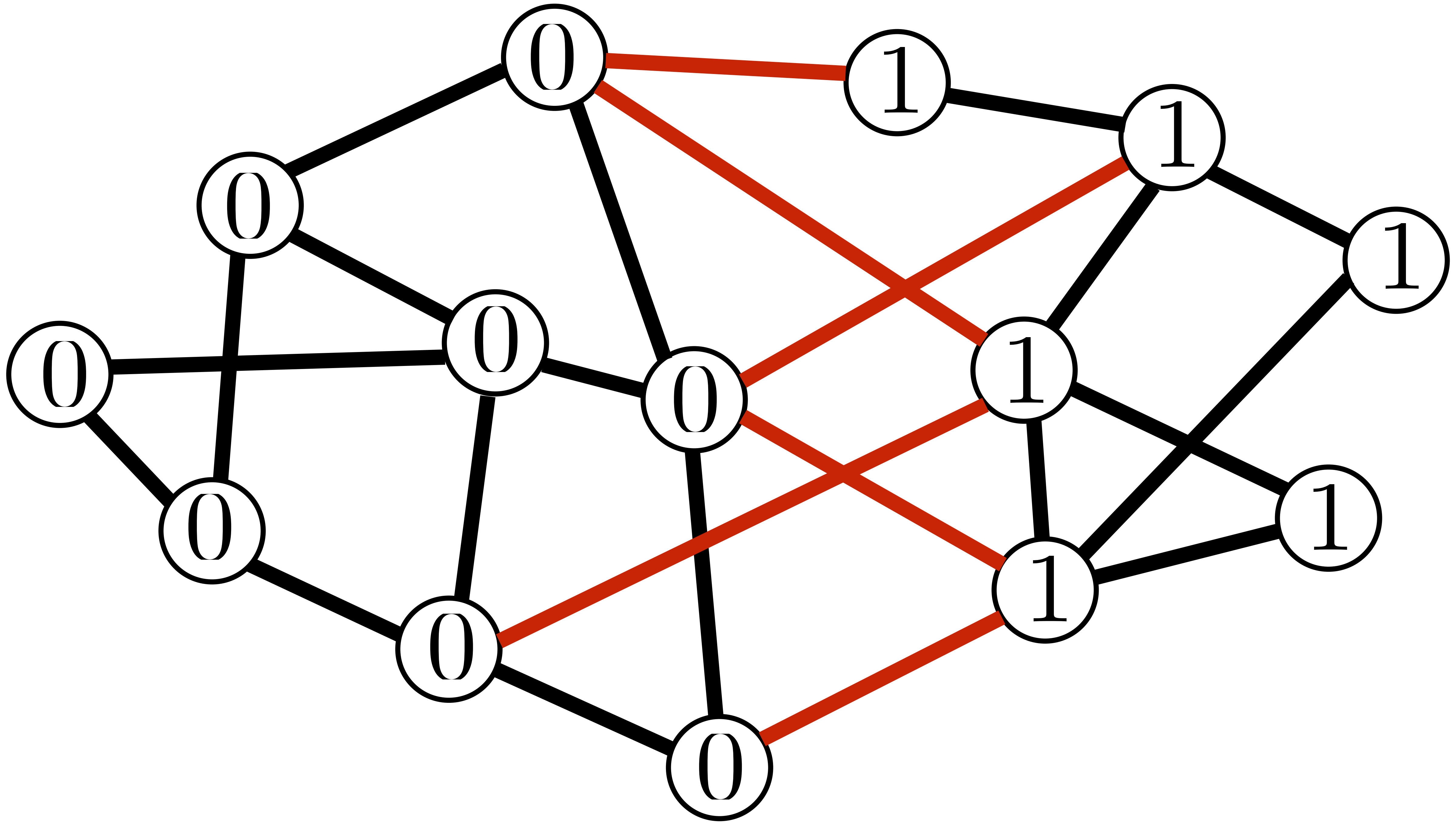


- APX-complete
- 0.878-approx

[Papadimitriou-Yannakakis JCSS'01]

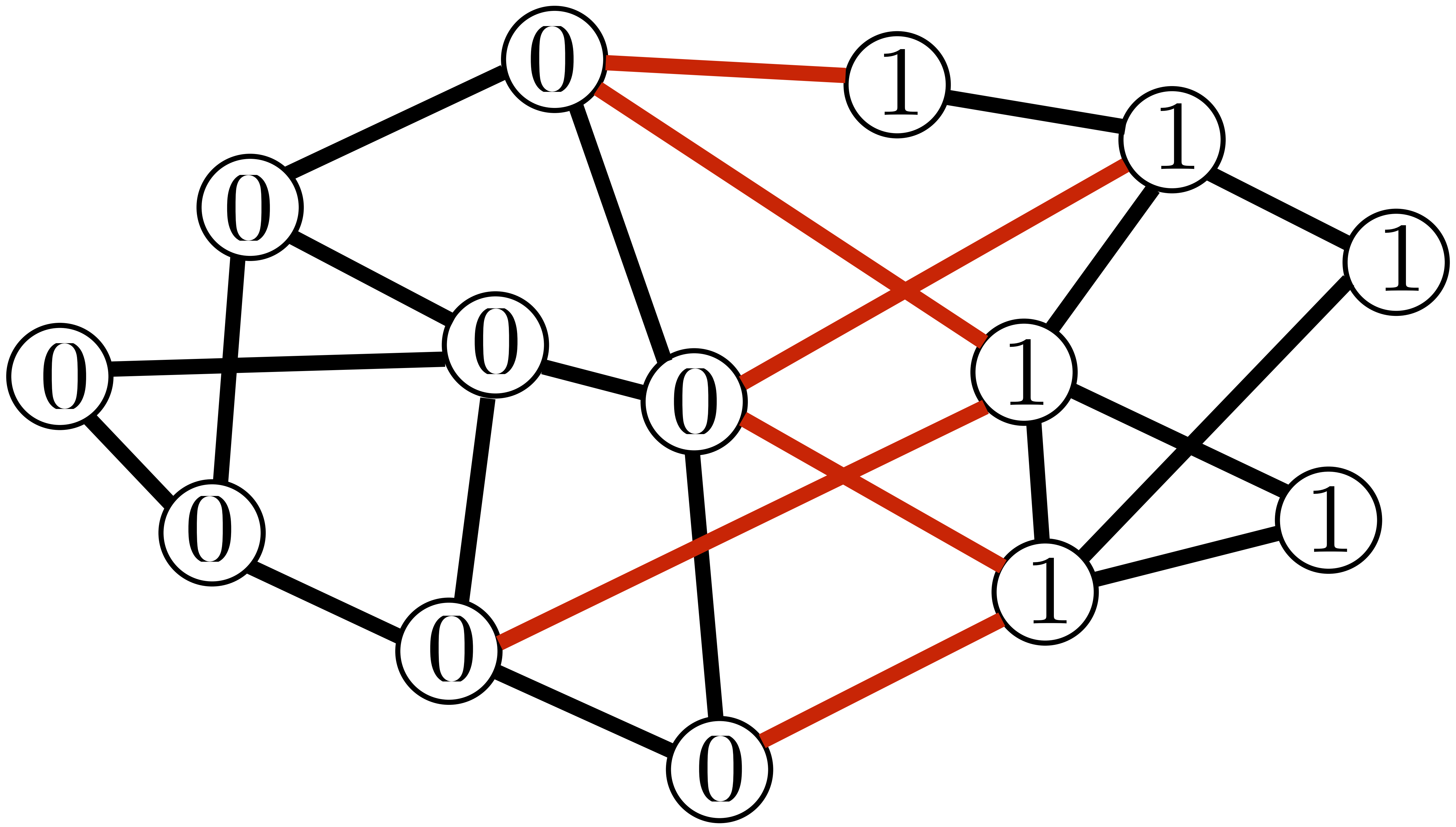
[Goemans-Williamson JACM'95]

Max-Cut



- APX-complete *[Papadimitriou-Yannakakis JCSS'01]*
- 0.878-approx *[Goemans-Williamson JACM'95]*
- 0.941-inapprox *[Trevisan et al. SICOMP'00]*

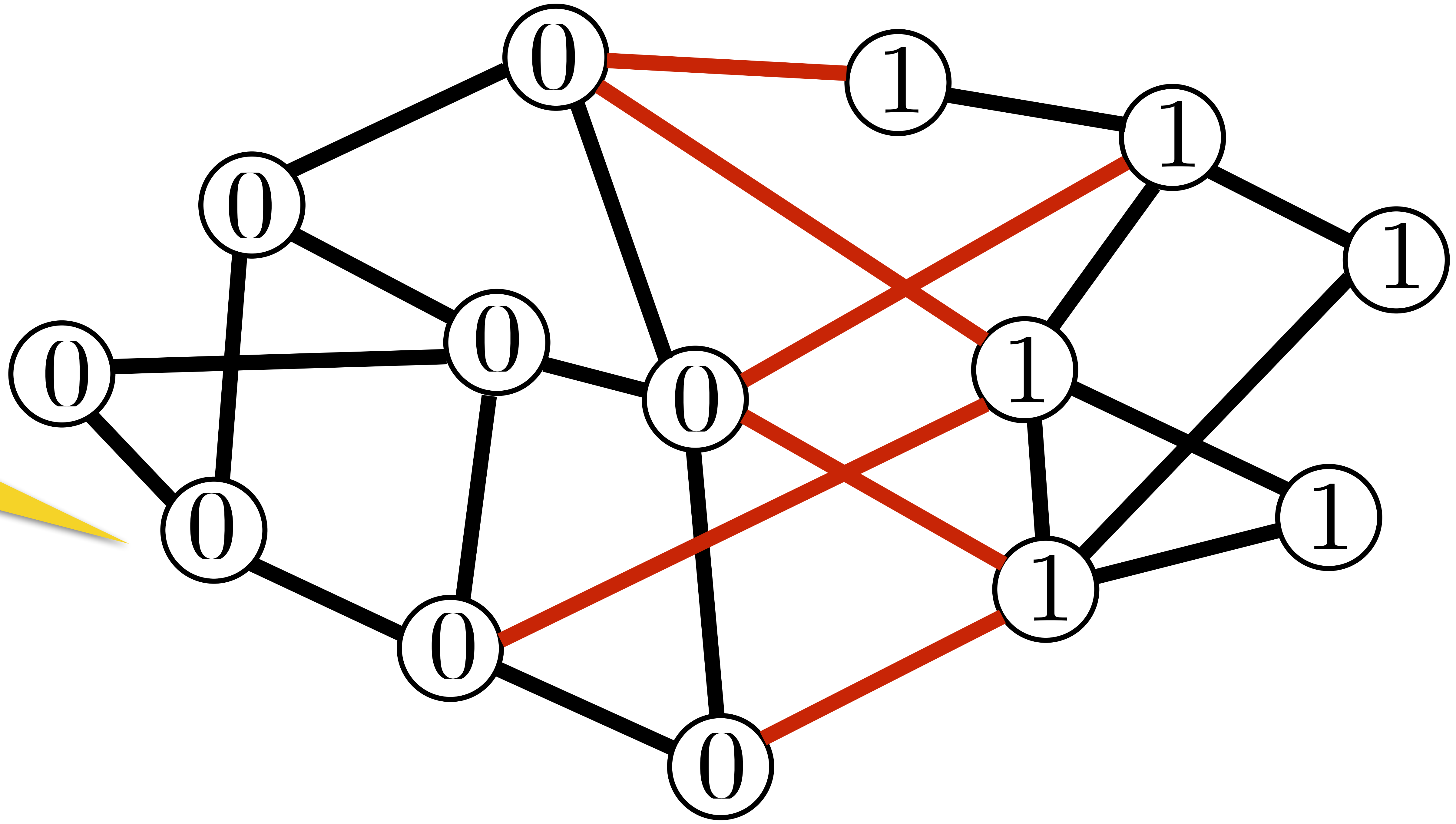
Max-Cut



Max-Cut

PTIME for planar

[Hadlock SICOMP'75]



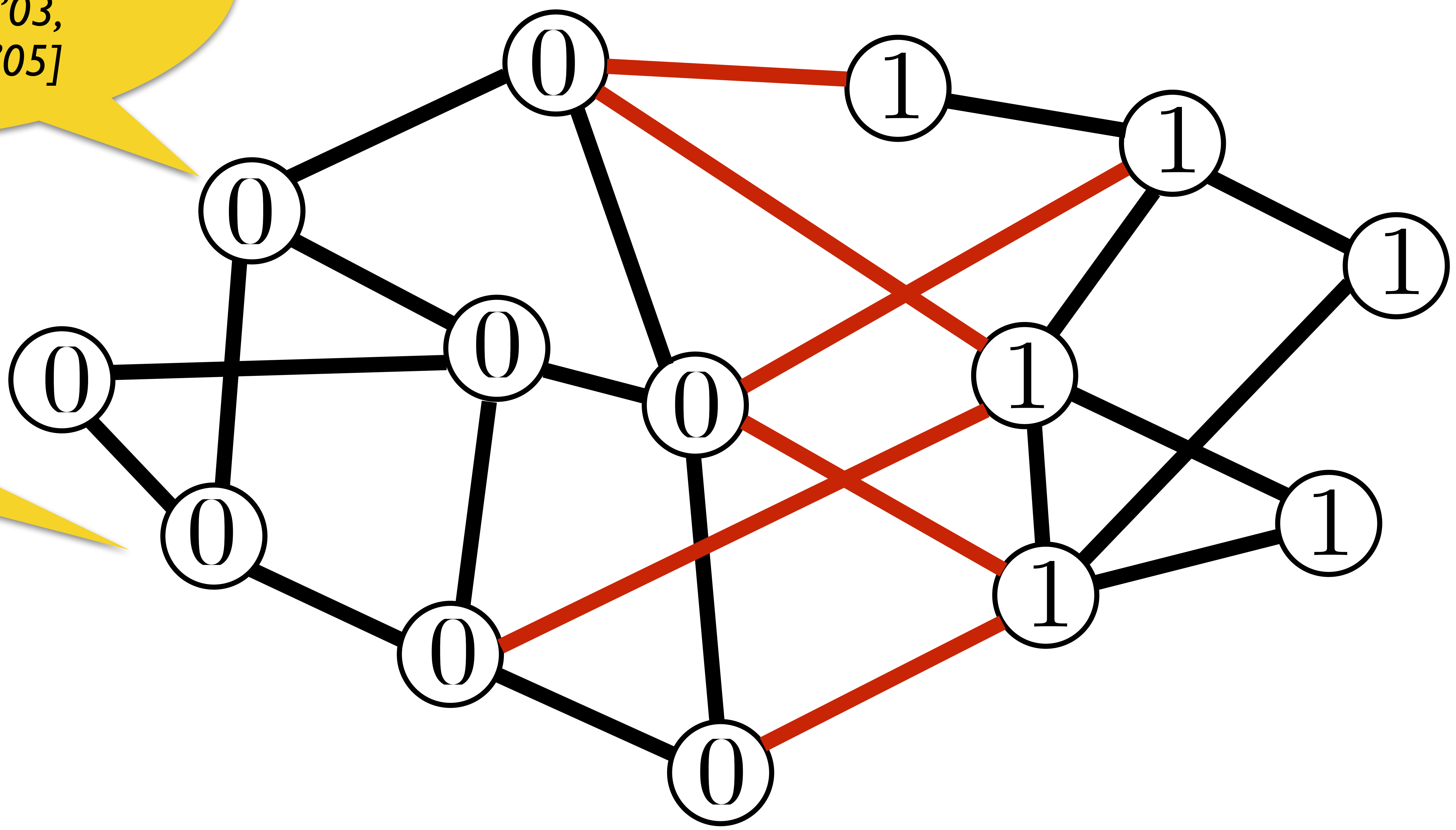
Max-Cut

PTAS for sparse

[Grohe Comb'03, Demaine et al. FOCS'05]

PTIME for planar

[Hadlock SICOMP'75]



Max-Cut

PTAS for sparse

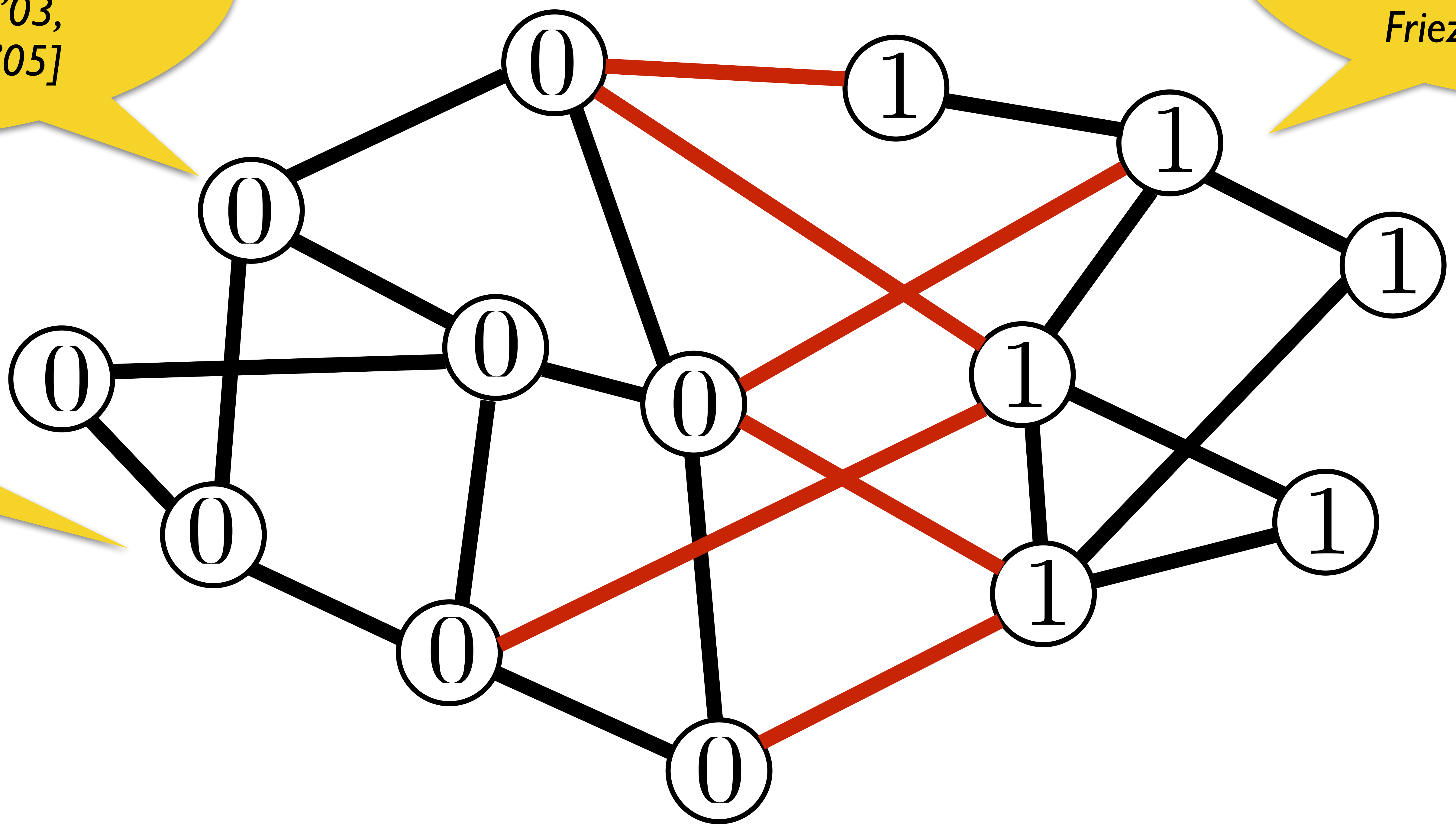
[Grohe Comb'03, Demaine et al. FOCS'05]

PTIME for planar

[Hadlock SICOMP'75]

PTAS for dense

[Arora et al. STOC'95, Frieze & Kannan FOCS'96]



Max-Cut

PTAS for sparse

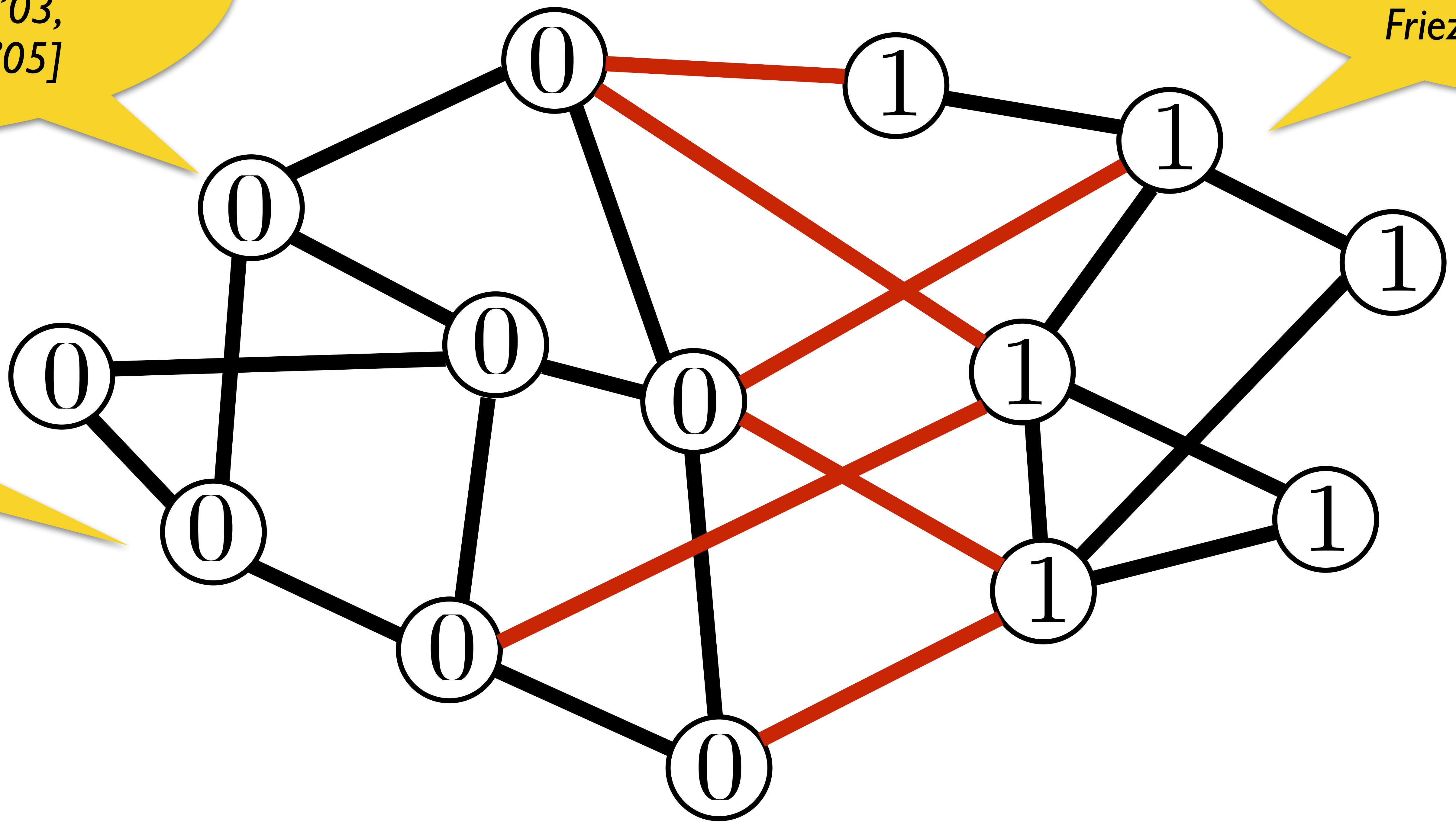
[Grohe Comb'03, Demaine et al. FOCS'05]

PTIME for planar

[Hadlock SICOMP'75]

PTAS for dense

[Arora et al. STOC'95, Frieze & Kannan FOCS'96]



What mathematical structure explains this?

Balázs Mezei

Miguel Romero

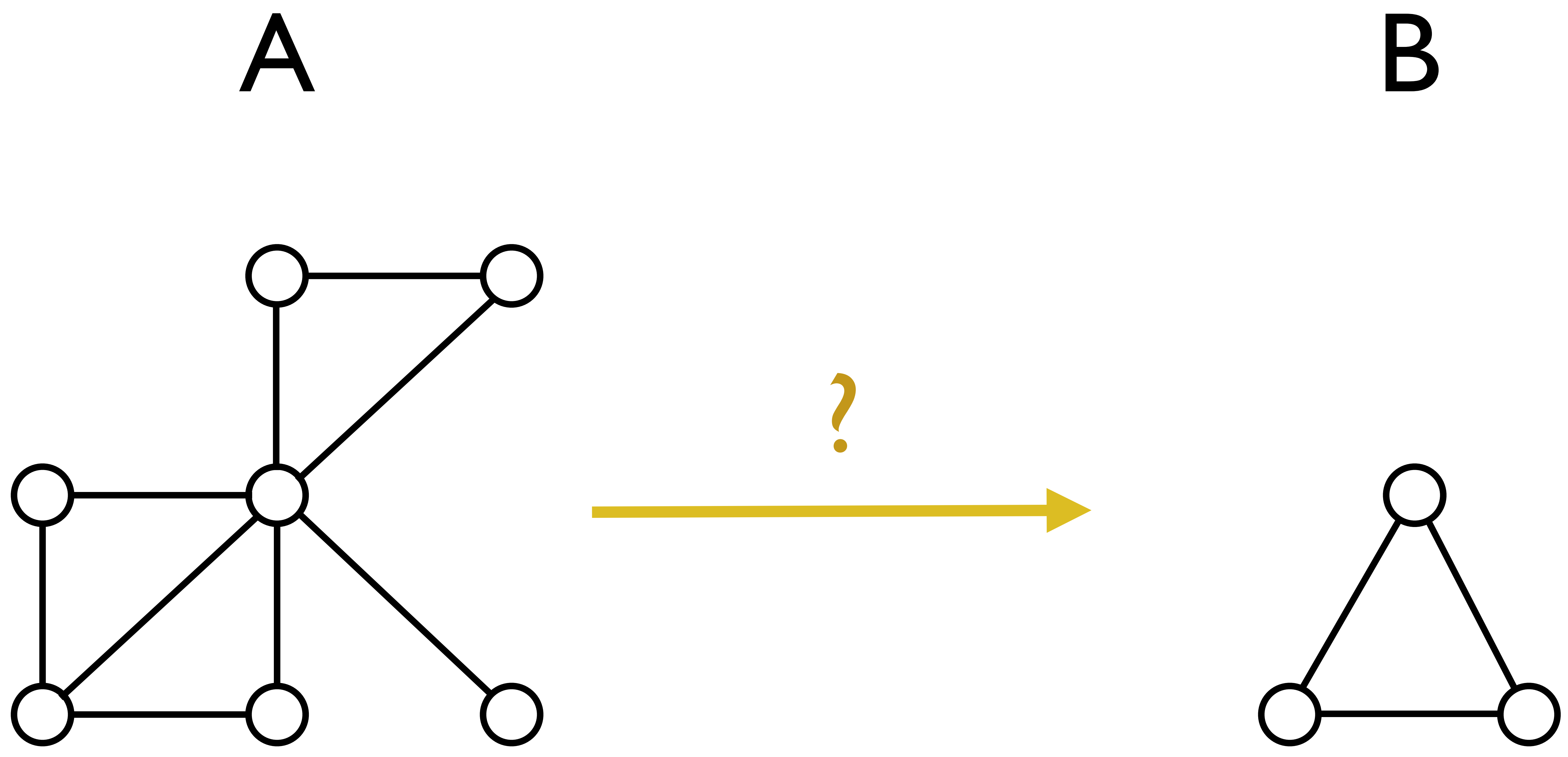
PUC

Marcin Wrochna

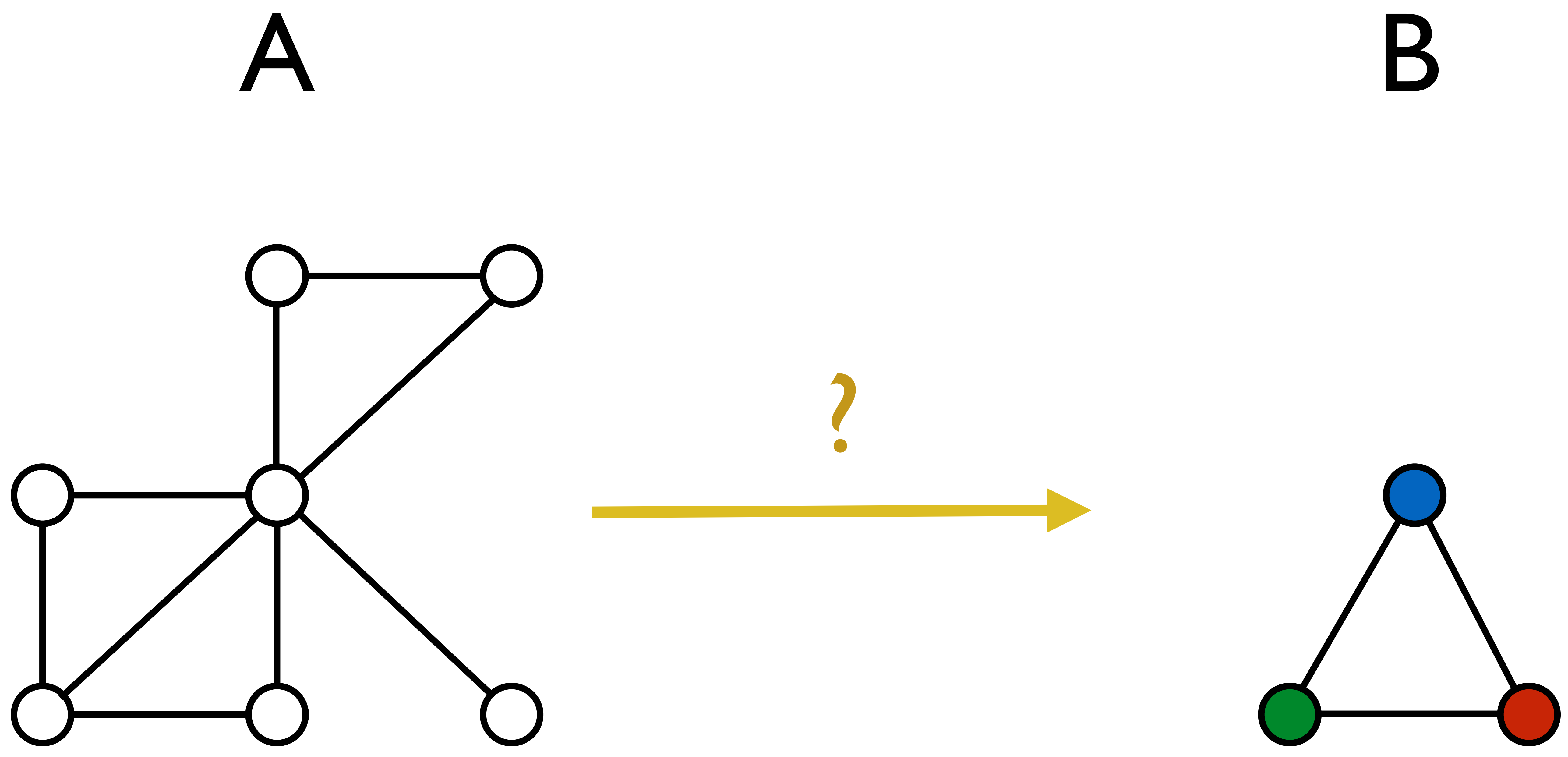
Warsaw

- Pliability and approximating MaxCSPs *[RWŽ SODA'21, JACM'23]*
- PTAS for general sparse general-valued CSPs *[MWŽ LICS'21, ACM TALG'23]*

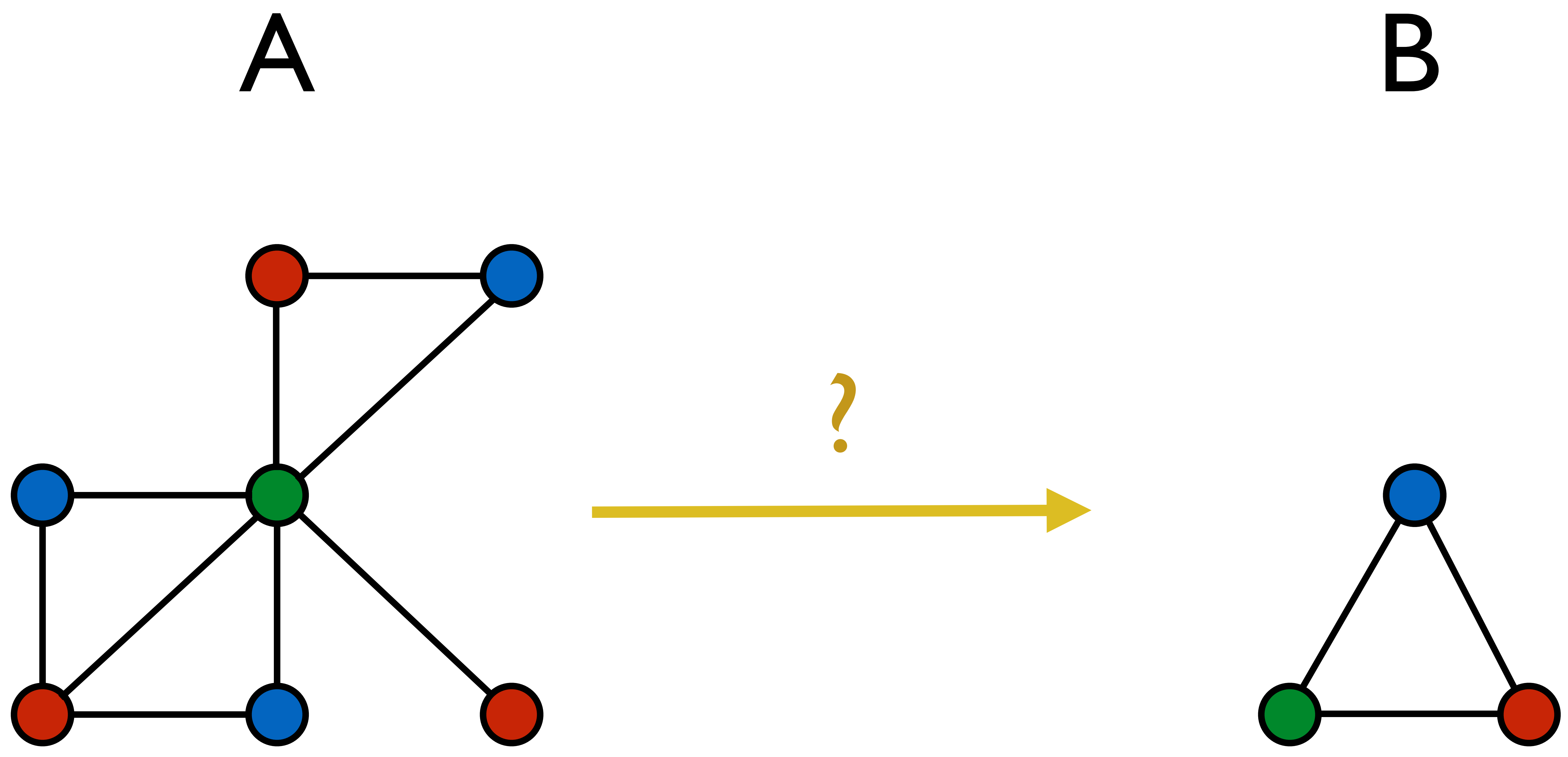
CSP



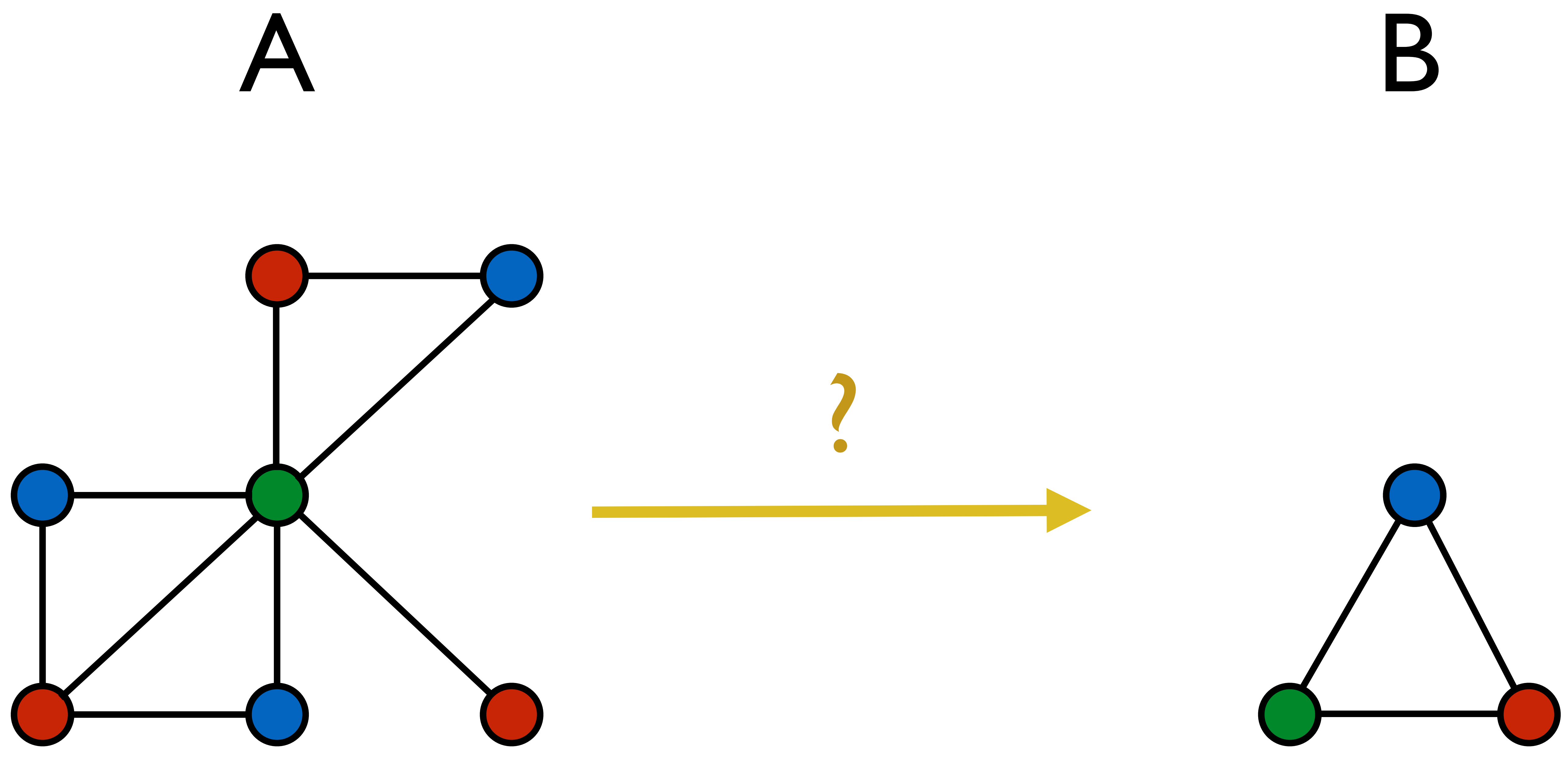
CSP



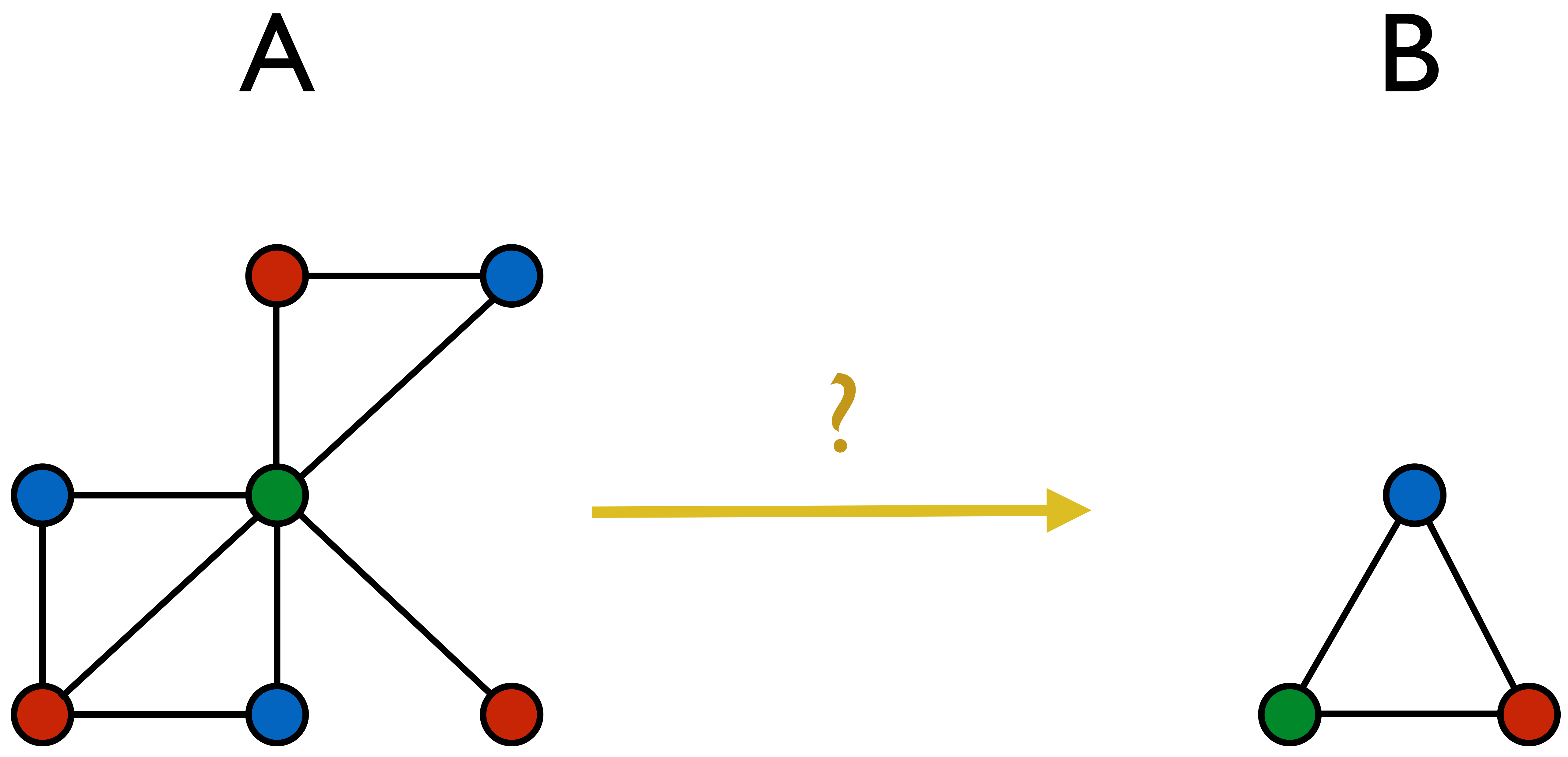
CSP



CSP(-, B)

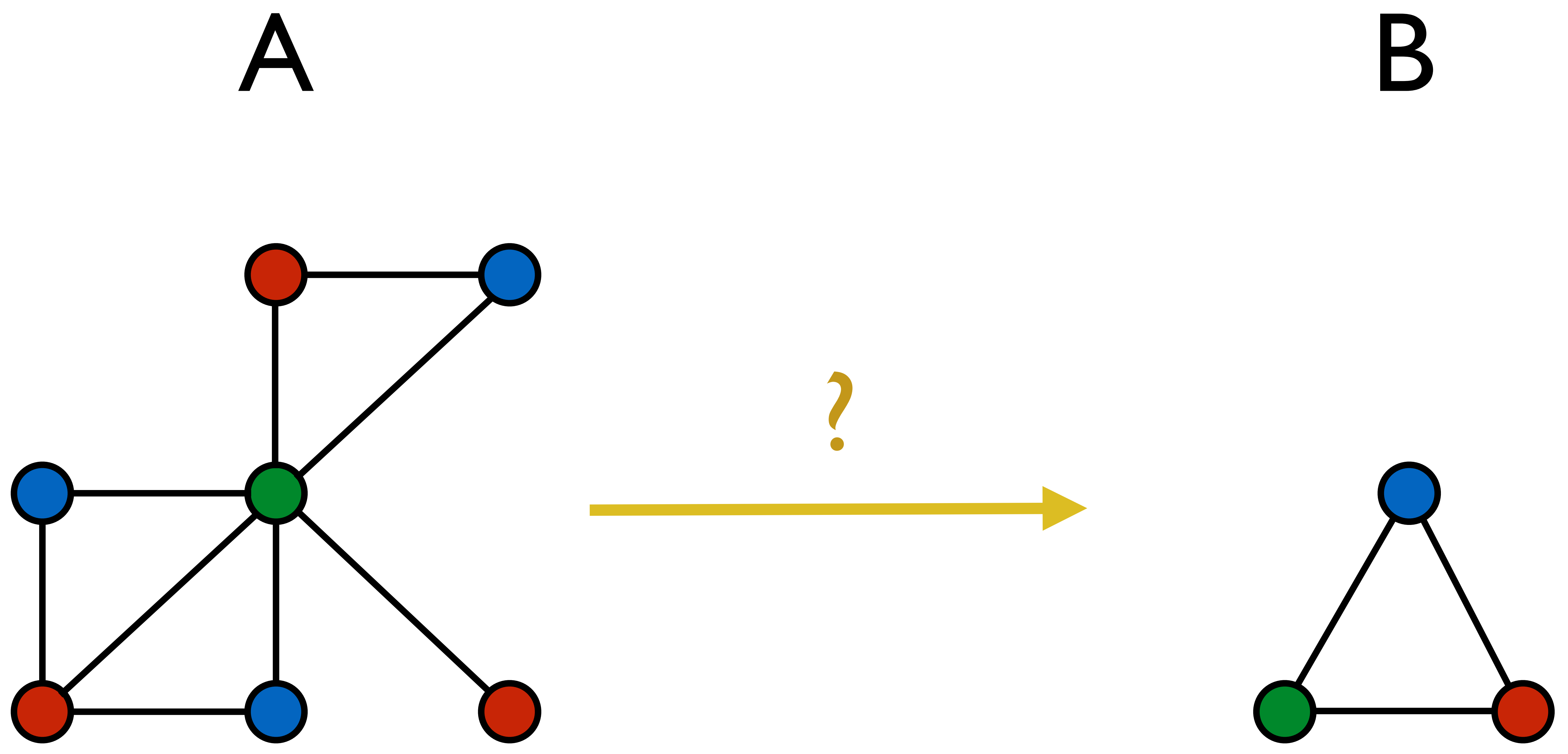


CSP(-, B)



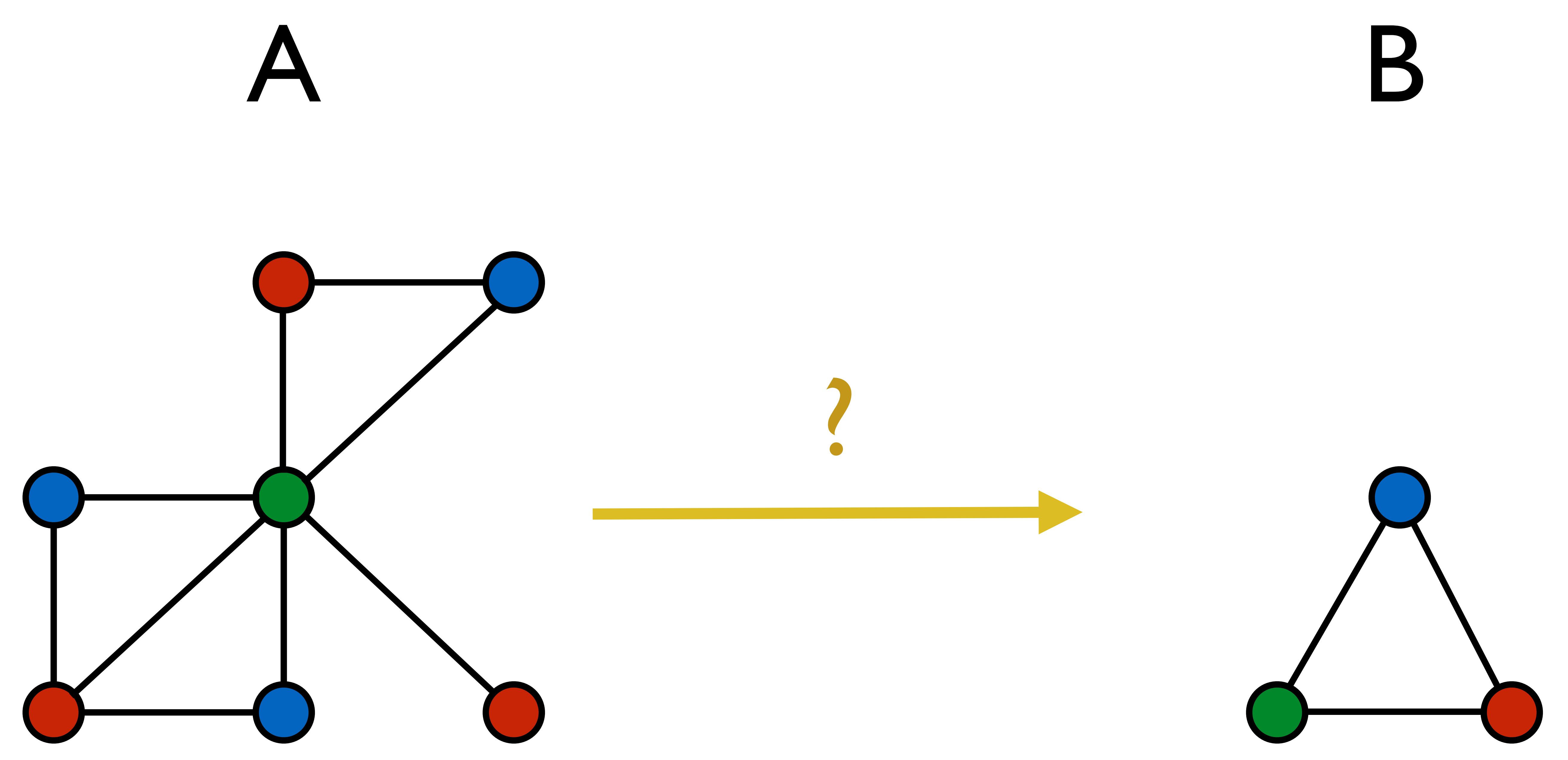
- CSP(-, H) ∈ PTIME or NP-complete *[Hell-Nešetřil JCTB'90]*

CSP(-, B)



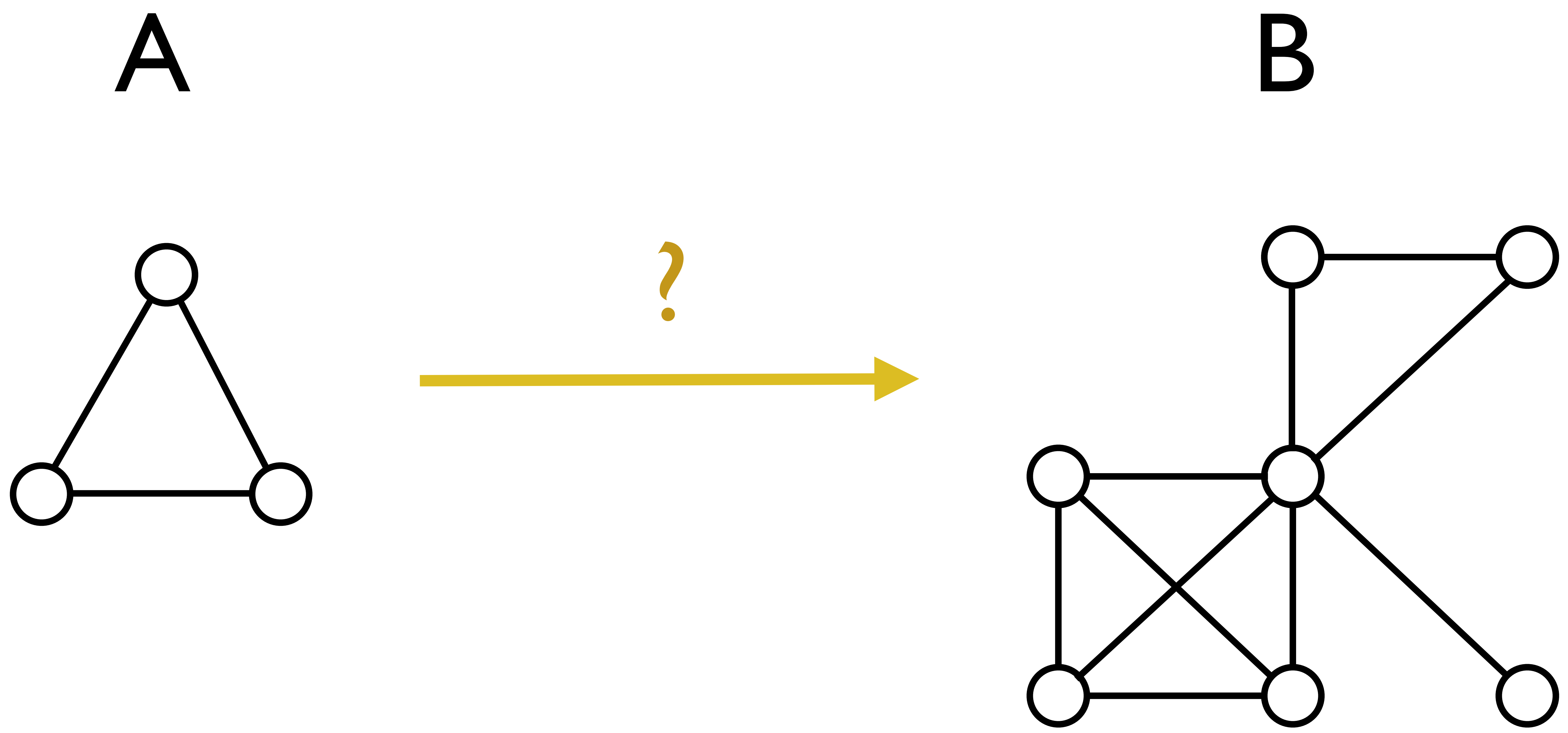
- CSP(-, H) ∈ PTIME or NP-complete *[Hell-Nešetřil JCTB'90]*
- CSP(-, B) ∈ PTIME or NP-complete *[Bulatov FOCS'17, Zhuk FOCS'17]*

CSP(-, B)

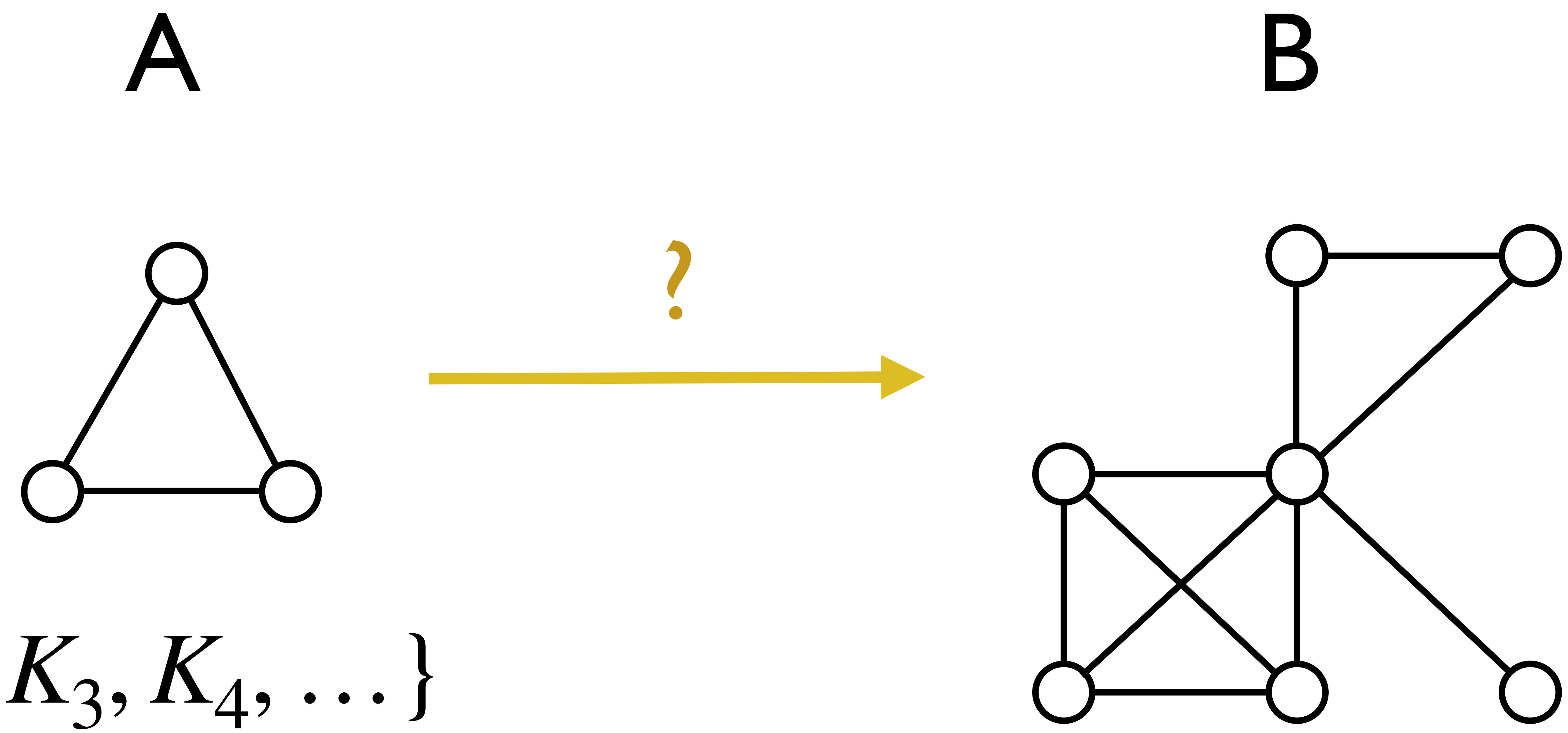


- CSP(-, H) ∈ PTIME or NP-complete [Hell-Nešetřil JCTB'90]
- CSP(-, B) ∈ PTIME or NP-complete [Bulatov FOCS'17, Zhuk FOCS'17]

CSP

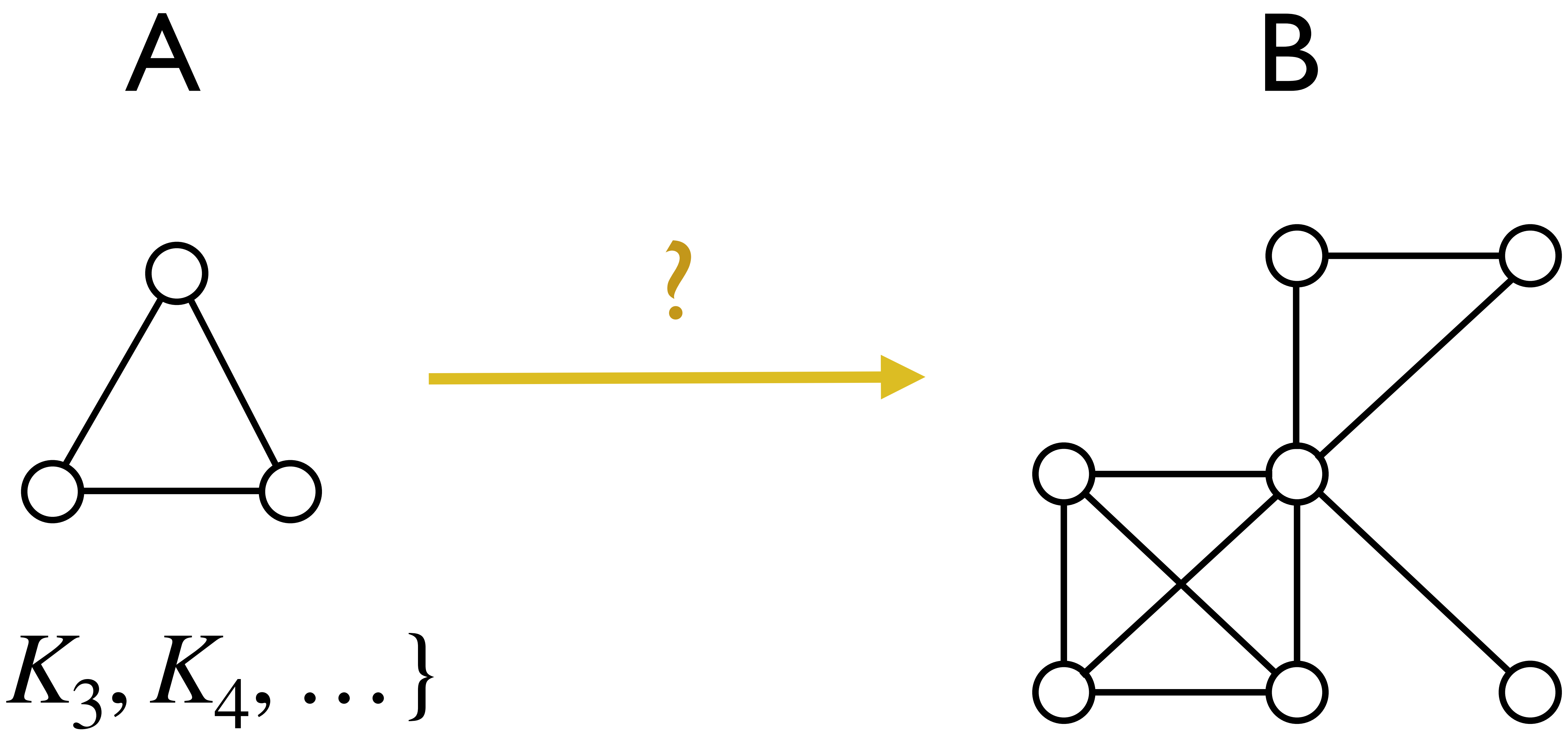


CSP($\mathscr{A}$, -)



$\mathscr{A} = \{K_3, K_4, \ldots\}$

CSP($\mathscr{A}$, -)

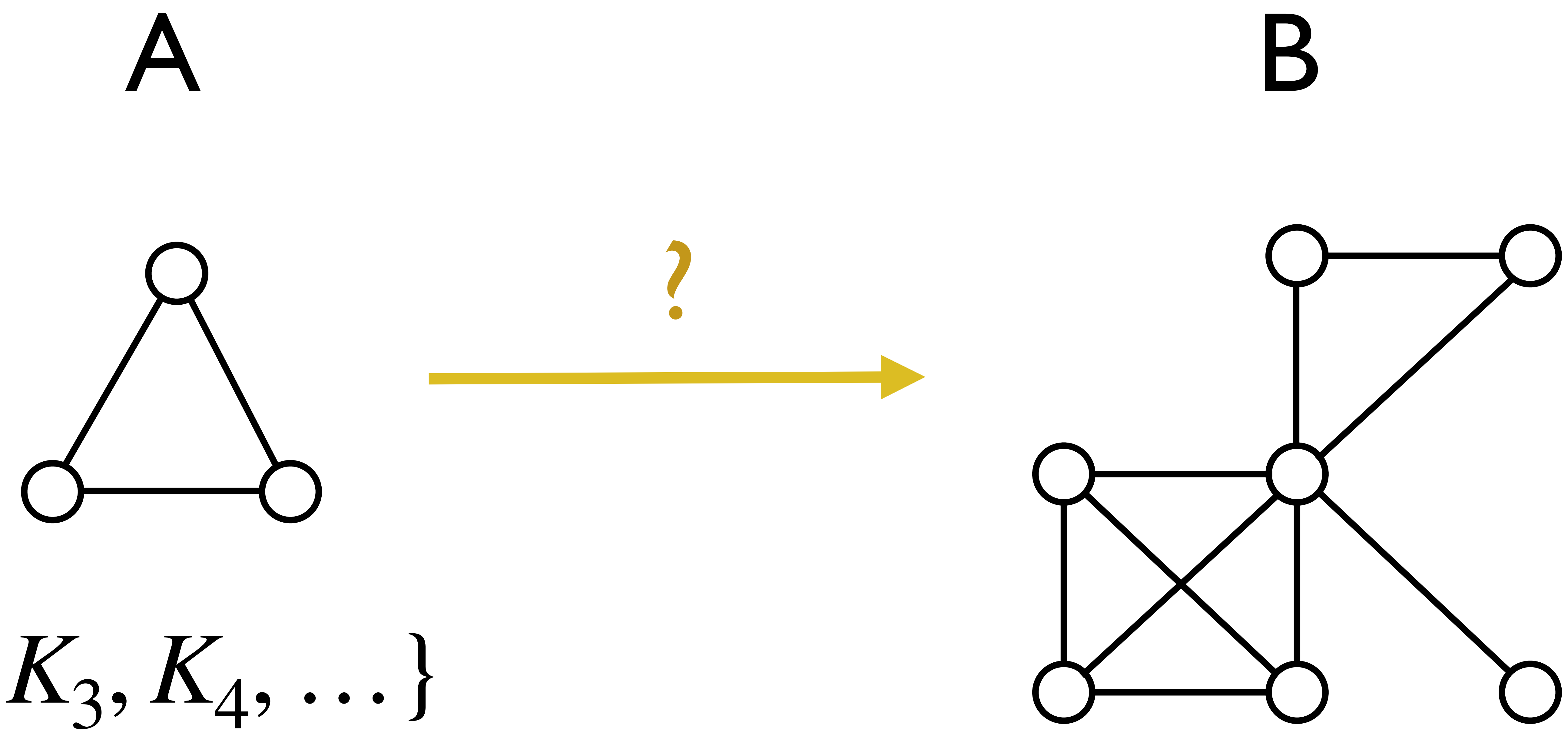


$$\mathscr{A} = \{K_3, K_4, \ldots\}$$

- CSP($\mathscr{A}$, -) $\in$ PTIME if tw($\mathscr{A}$) bounded

[Freuder AAAI'90]

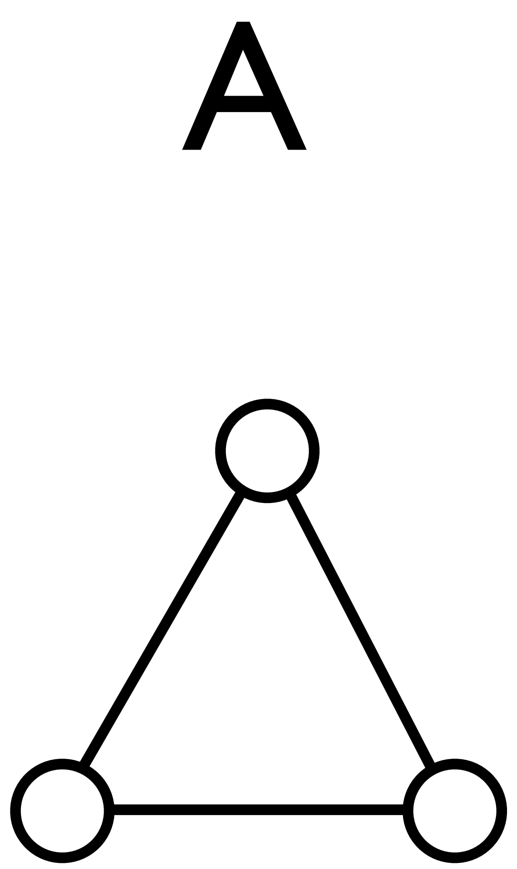
CSP($\mathscr{A}$, -)

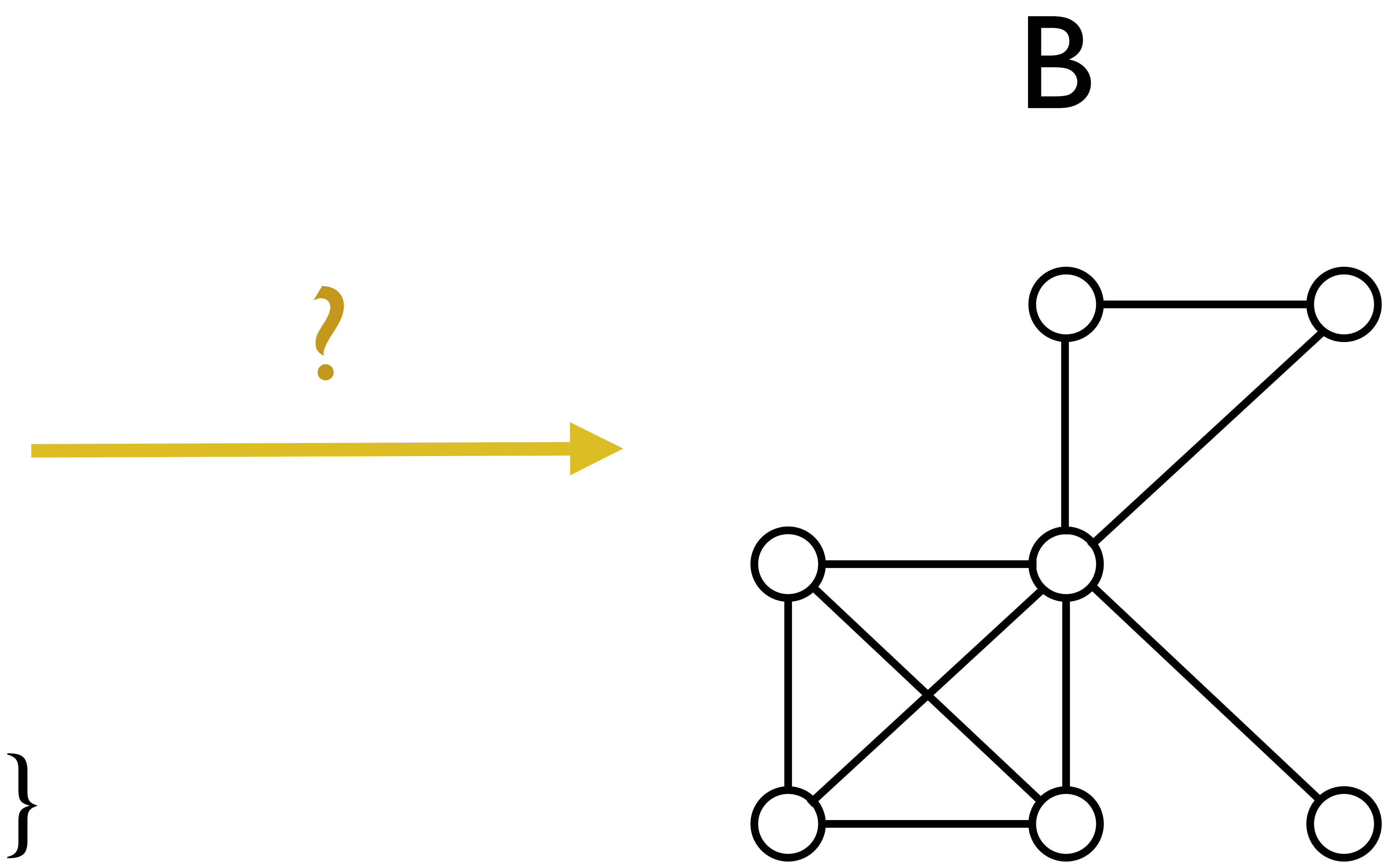


$\mathscr{A} = \{K_3, K_4, \ldots\}$

- CSP($\mathscr{A}$, -) $\in$ PTIME if tw($\mathscr{A}$) bounded *[Freuder AAAI'90]*
- CSP($\mathscr{A}_{\mathscr{G}}$, -) $\notin$ PTIME if tw($\mathscr{G}$) unbounded *[Grohe-Schwentick-Segoufin STOC'01]*

CSP($\mathscr{A}$, -)





FPT≠W[1]

$$\mathscr{A} = \{K_3, K_4, \ldots\}$$

- CSP($\mathscr{A}$, -) $\in$ PTIME if tw($\mathscr{A}$) bounded *[Freuder AAAI'90]*
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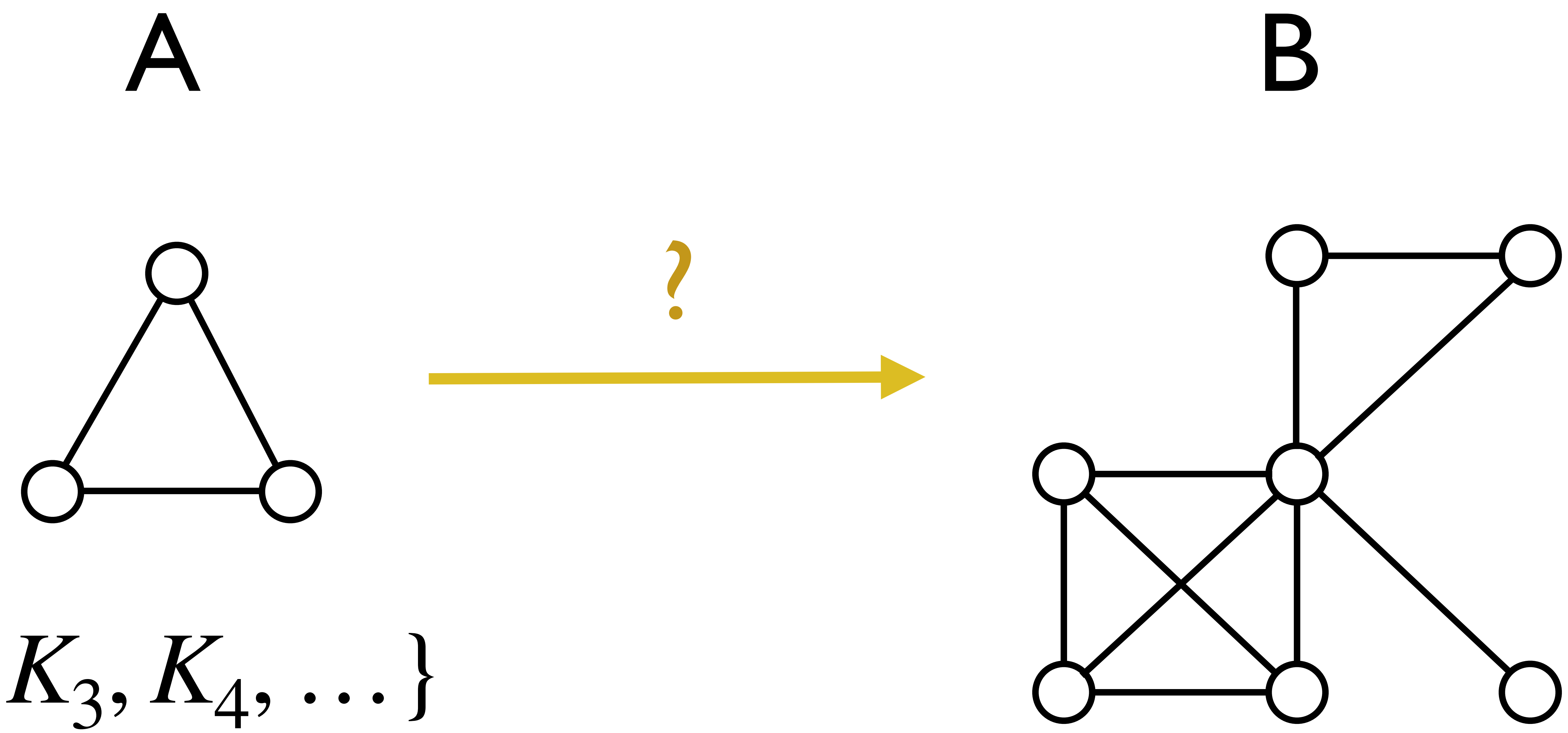

FPT≠W[1]

$\mathscr{A} = \{K_3, K_4, \ldots\}$

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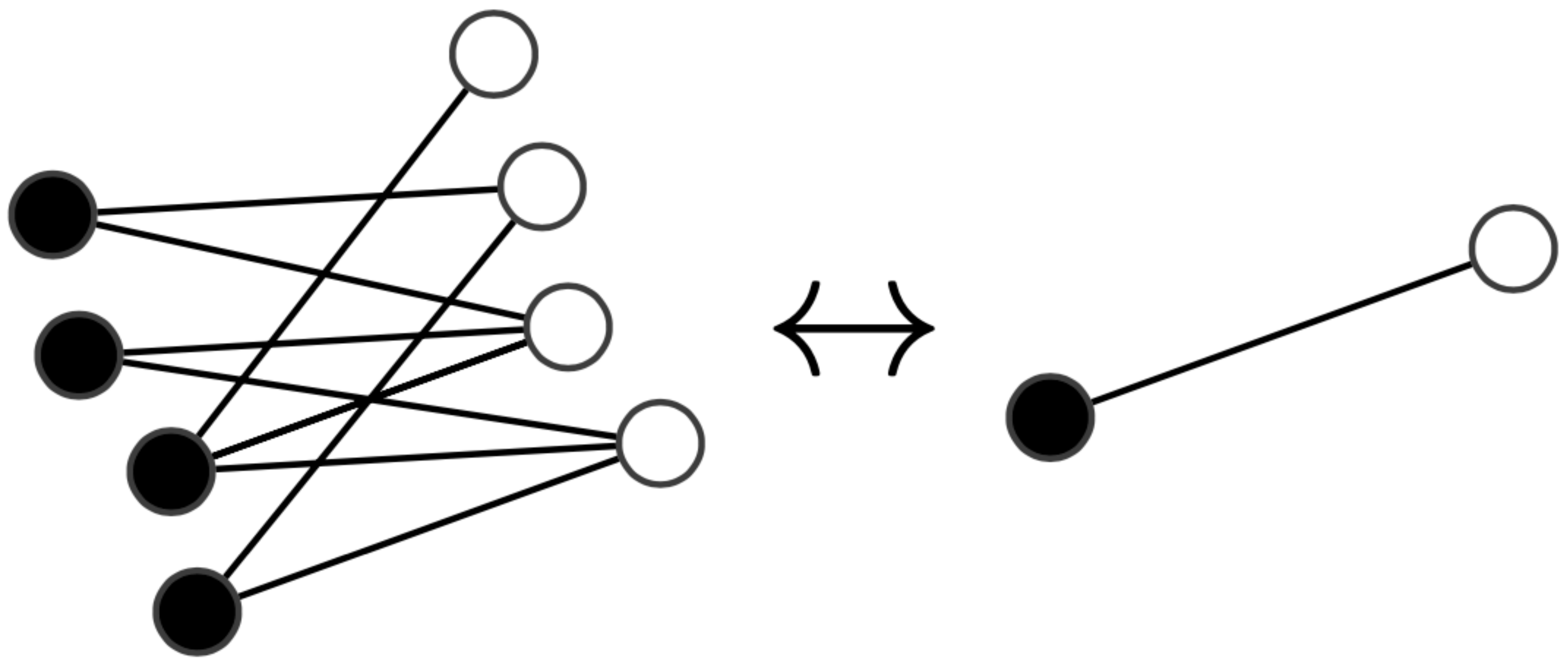
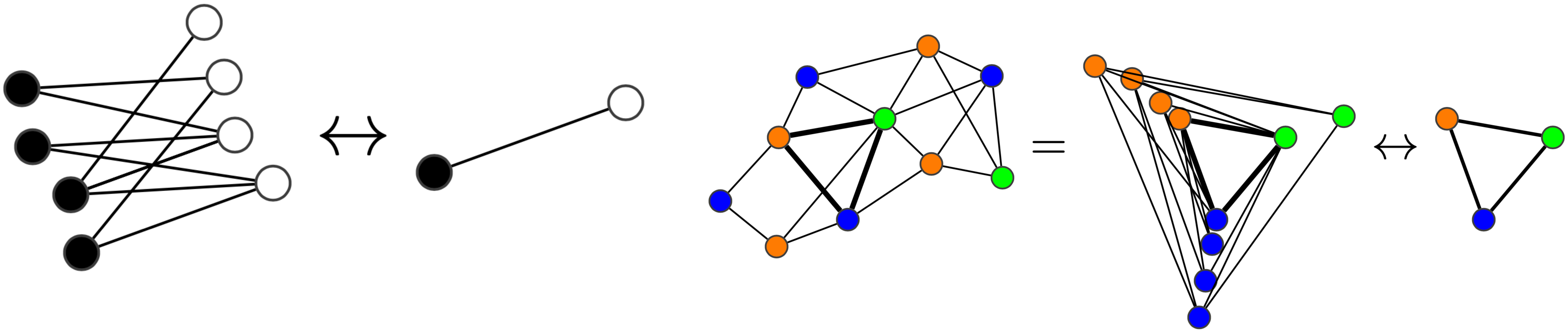
- CSP($\mathscr{A}_{\mathscr{G}}$, -) $\notin$ PTIME if tw($\mathscr{G}$) unbounded *[Grohe-Schwentick-Segoufin STOC'01]*
- CSP($\mathscr{A}$, -) $\in$ PTIME if tw(core($\mathscr{A}$)) bounded *[Dalmau-Kolaitis-Vardi CP'02]*

CSP($\mathscr{A}$, -)



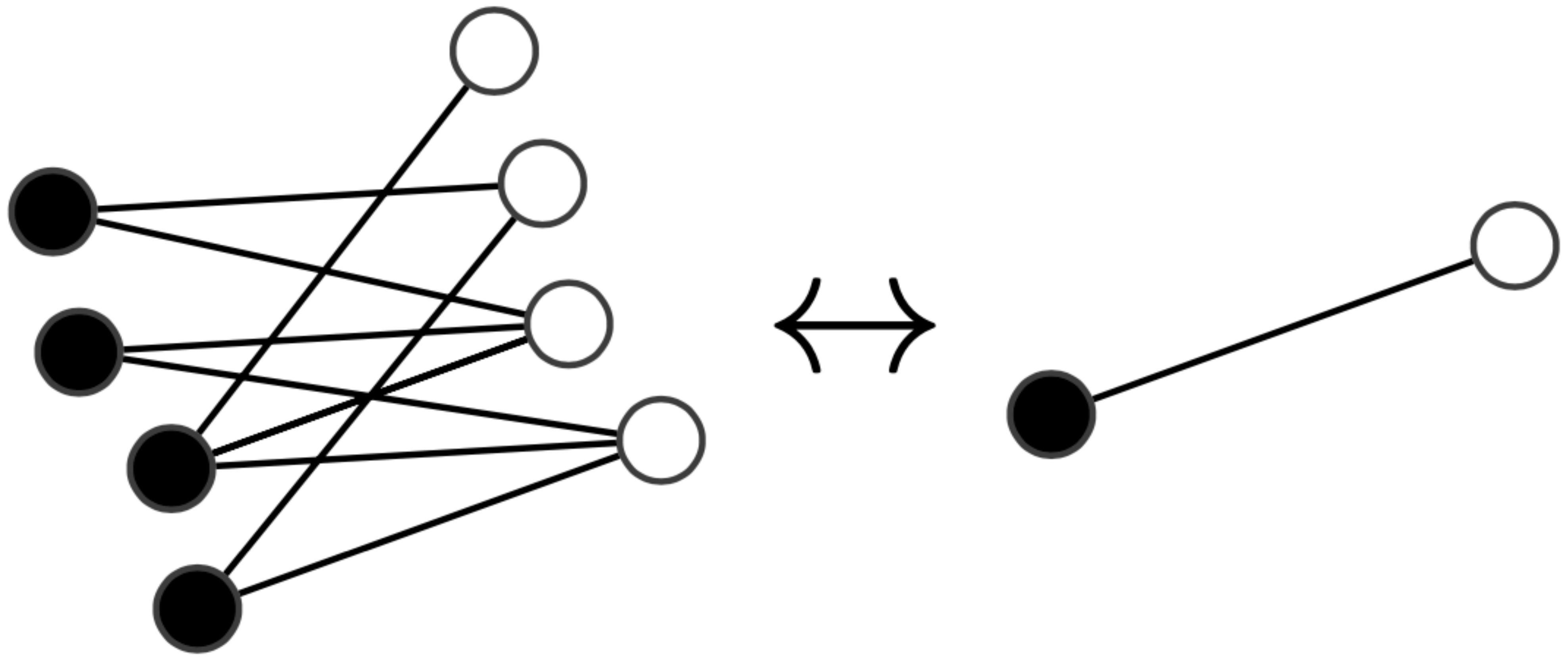
- CSP($\mathscr{A}$, -) $\in$ PTIME if tw($\mathscr{A}$) bounded *[Freuder AAAI'90]*
- CSP($\mathscr{A}_{\mathscr{G}}$, -) $\notin$ PTIME if tw($\mathscr{G}$) unbounded *[Grohe-Schwentick-Segoufin STOC'01]*
- CSP($\mathscr{A}$, -) $\in$ PTIME if tw(core($\mathscr{A}$)) bounded *[Dalmau-Kolaitis-Vardi CP'02]*

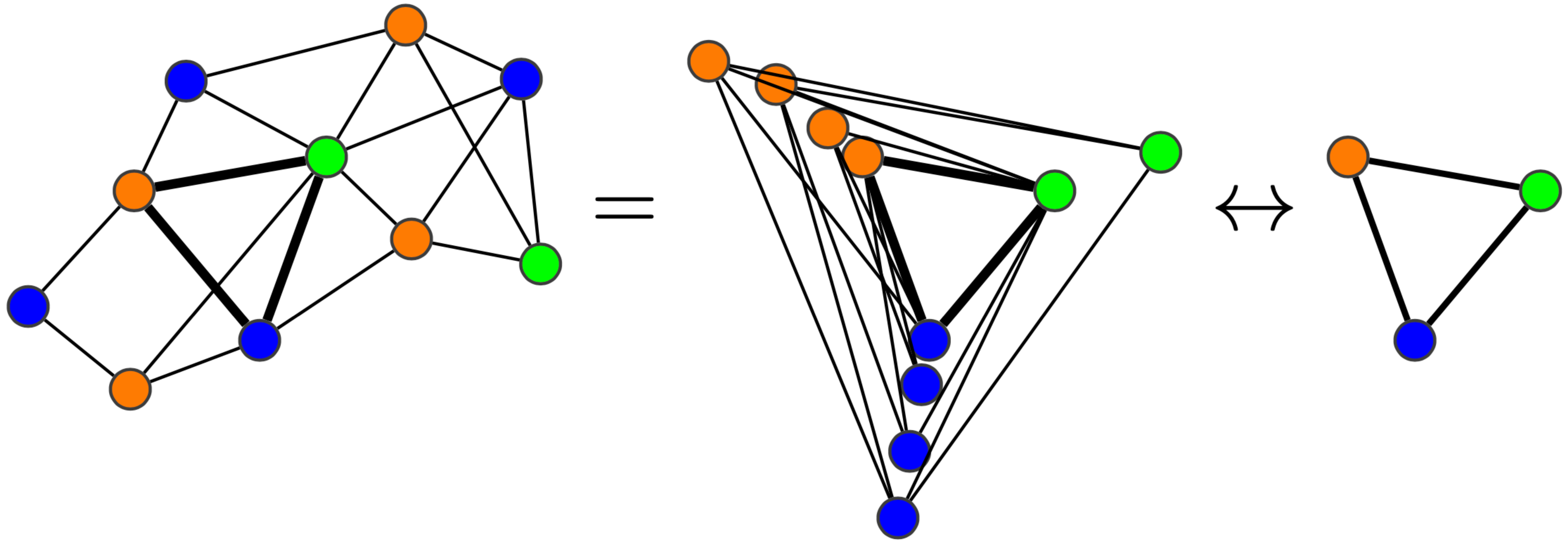
CSP($\mathscr{A}$, -)



- CSP($\mathscr{A}$, -) $\in$ PTIME if tw($\mathscr{A}$) bounded *[Freuder AAAI'90]*
- CSP($\mathscr{A}_{\mathscr{G}}$, -) $\notin$ PTIME if tw($\mathscr{G}$) unbounded *[Grohe-Schwentick-Segoufin STOC'01]*
- CSP($\mathscr{A}$, -) $\in$ PTIME if tw(core($\mathscr{A}$)) bounded *[Dalmau-Kolaitis-Vardi CP'02]*

CSP($\mathscr{A}$, -)





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- CSP($\mathscr{A}_{\mathscr{G}}$, -) $\notin$ PTIME if tw($\mathscr{G}$) unbounded *[Grohe-Schwentick-Segoufin STOC'01]*
- CSP($\mathscr{A}$, -) $\in$ PTIME if tw(core($\mathscr{A}$)) bounded *[Dalmau-Kolaitis-Vardi CP'02]*
- CSP($\mathscr{A}$, -) $\notin$ PTIME if tw(core($\mathscr{A}$)) unbounded *[Grohe JACM'07]*

The Complexity of Homomorphism and Constraint Satisfaction Problems Seen from the Other Side

MARTIN GROHE

Humboldt-Universität zu Berlin, Berlin, Germany

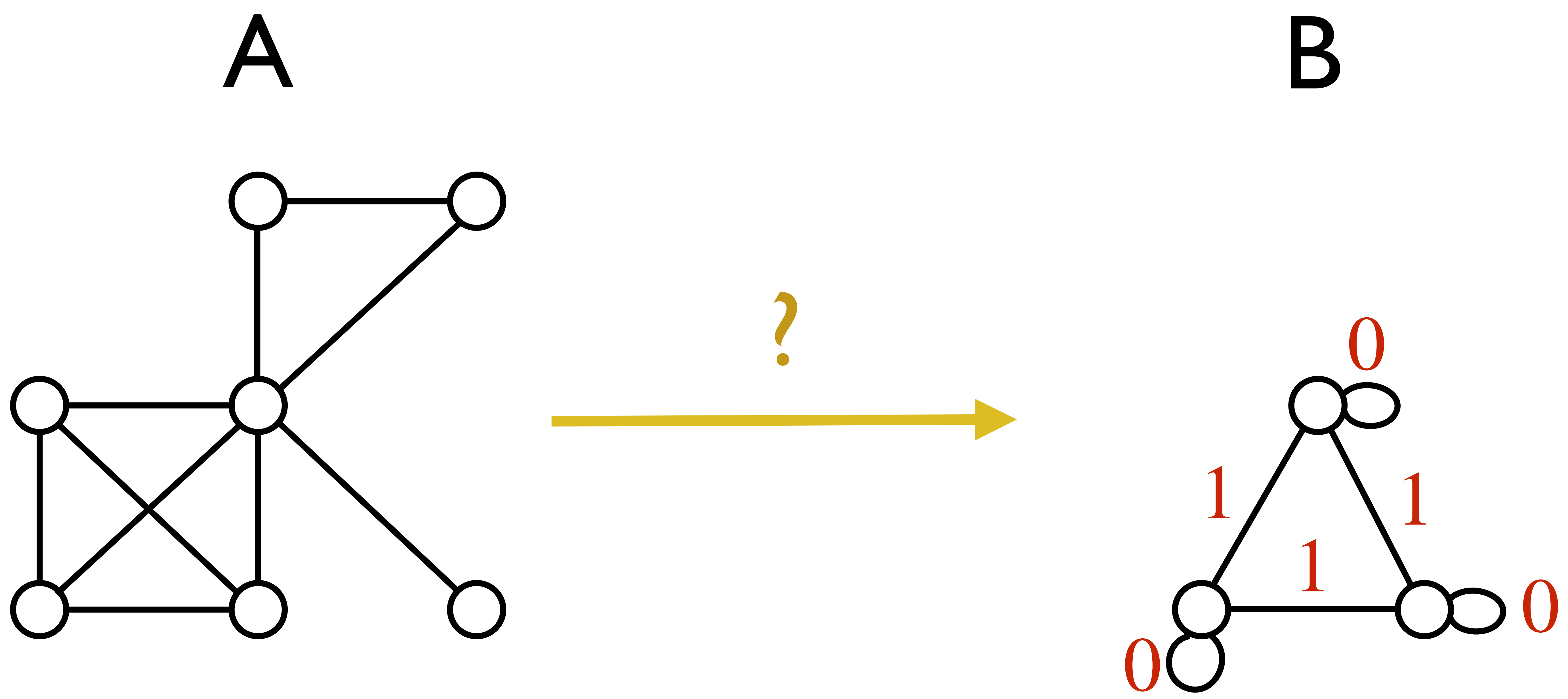
Abstract. We give a complexity theoretic classification of homomorphism problems for graphs and, more generally, relational structures obtained by restricting the left hand side structure in a homomorphism. For every class C of structures, let HOM(C, −) be the problem of deciding whether a given structure $\mathcal{A} \in$ C has a homomorphism to a given (arbitrary) structure $\mathcal{B}$. We prove that, under some complexity theoretic assumption from parameterized complexity theory, HOM(C, −) is in polynomial time if and only if C has bounded tree width modulo homomorphic equivalence.

Translated into the language of constraint satisfaction problems, our result yields a characterization of the tractable structural restrictions of constraint satisfaction problems. Translated into the language of database theory, it implies a characterization of the tractable instances of the evaluation problem for conjunctive queries over relational databases.

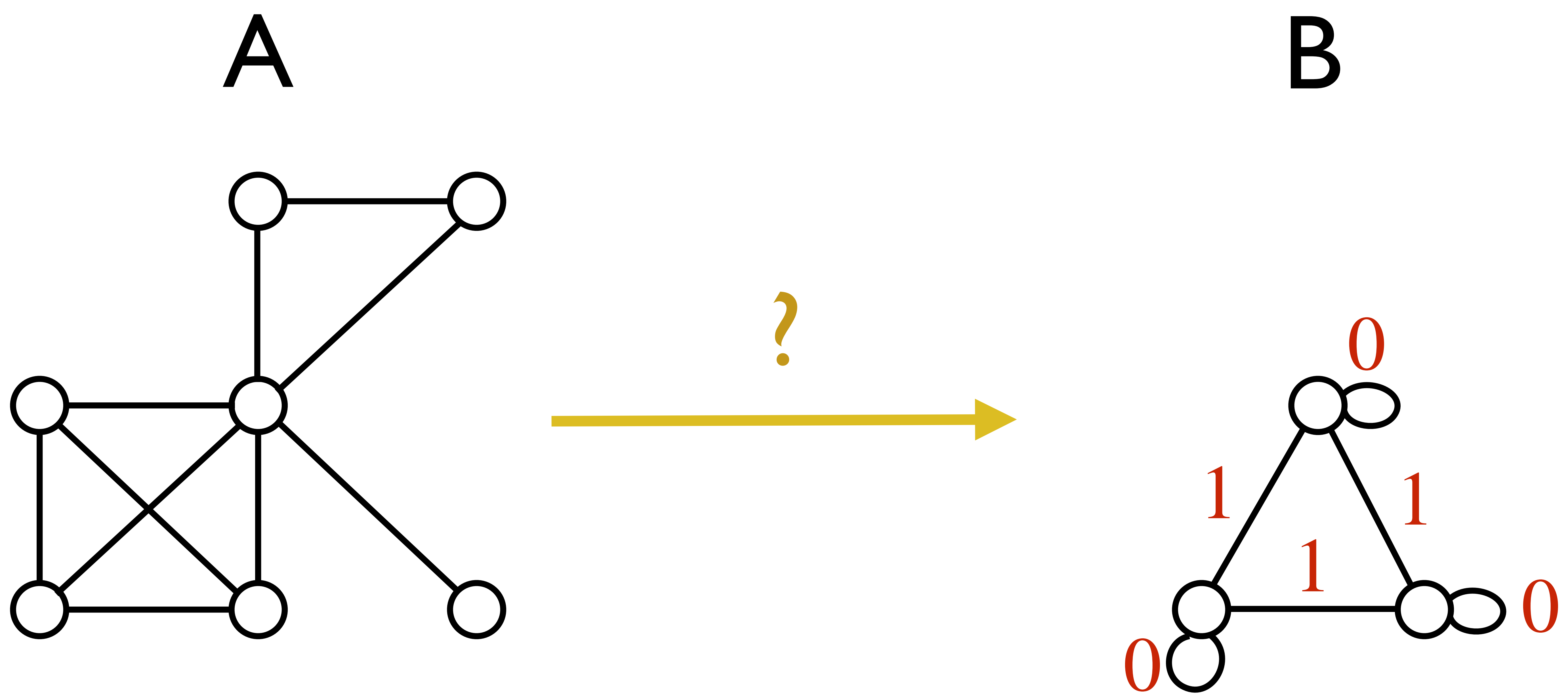
- CSP($\mathscr{A}$, -) $\notin$ PTIME if tw(core($\mathscr{A}$)) unbounded

[Grohe JACM'07]

MaxCSP(-, B)



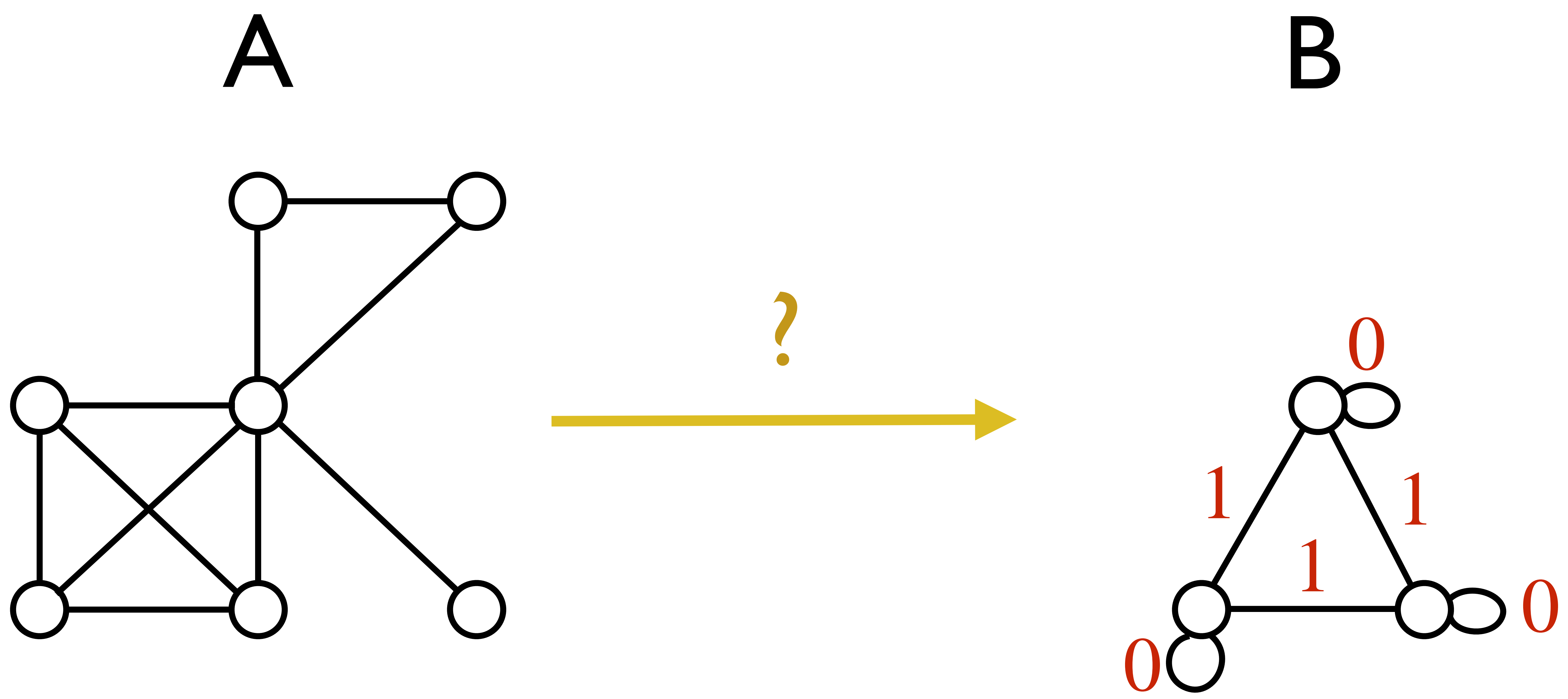
MaxCSP(-, B)



- MaxCSP(-, H) ∈ PTIME or NP-complete

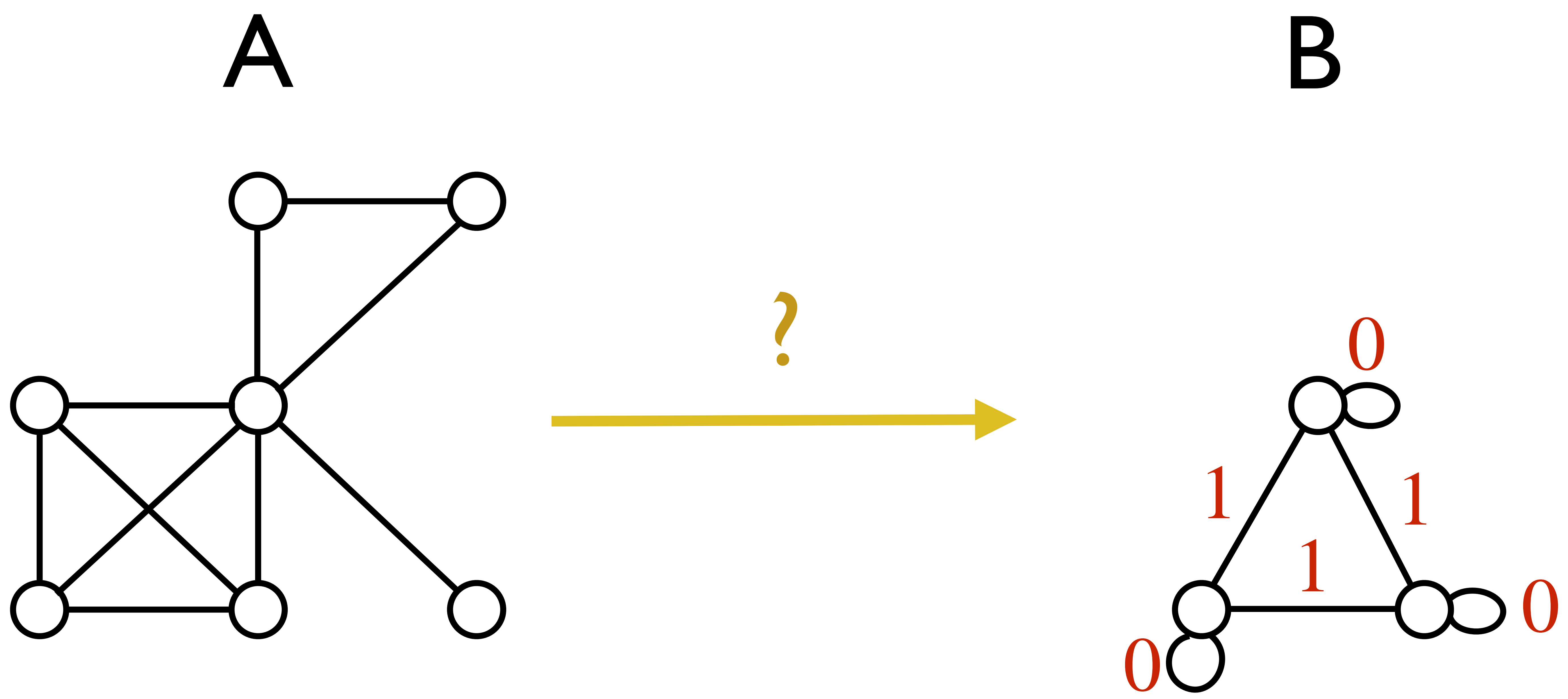
[Jonsson-Krokhin JCSS'07]

MaxCSP(-, B)



- MaxCSP(-, H) ∈ PTIME or NP-complete *[Jonsson-Krokhin JCSS'07]*
- MaxCSP(-, B) ∈ PTIME or NP-complete *[Thapper-**Ž.** JACM'16]*

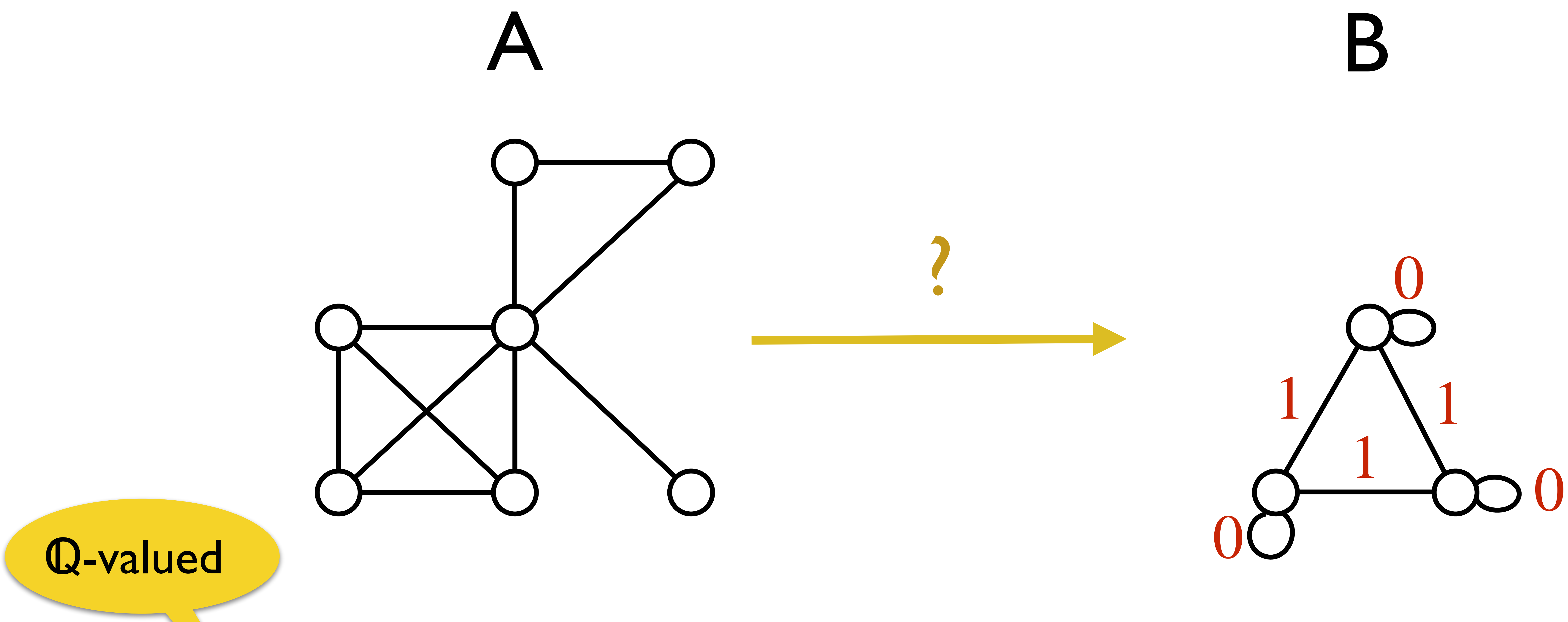
MaxCSP(-, B)



$\mathbb{Q}$-valued

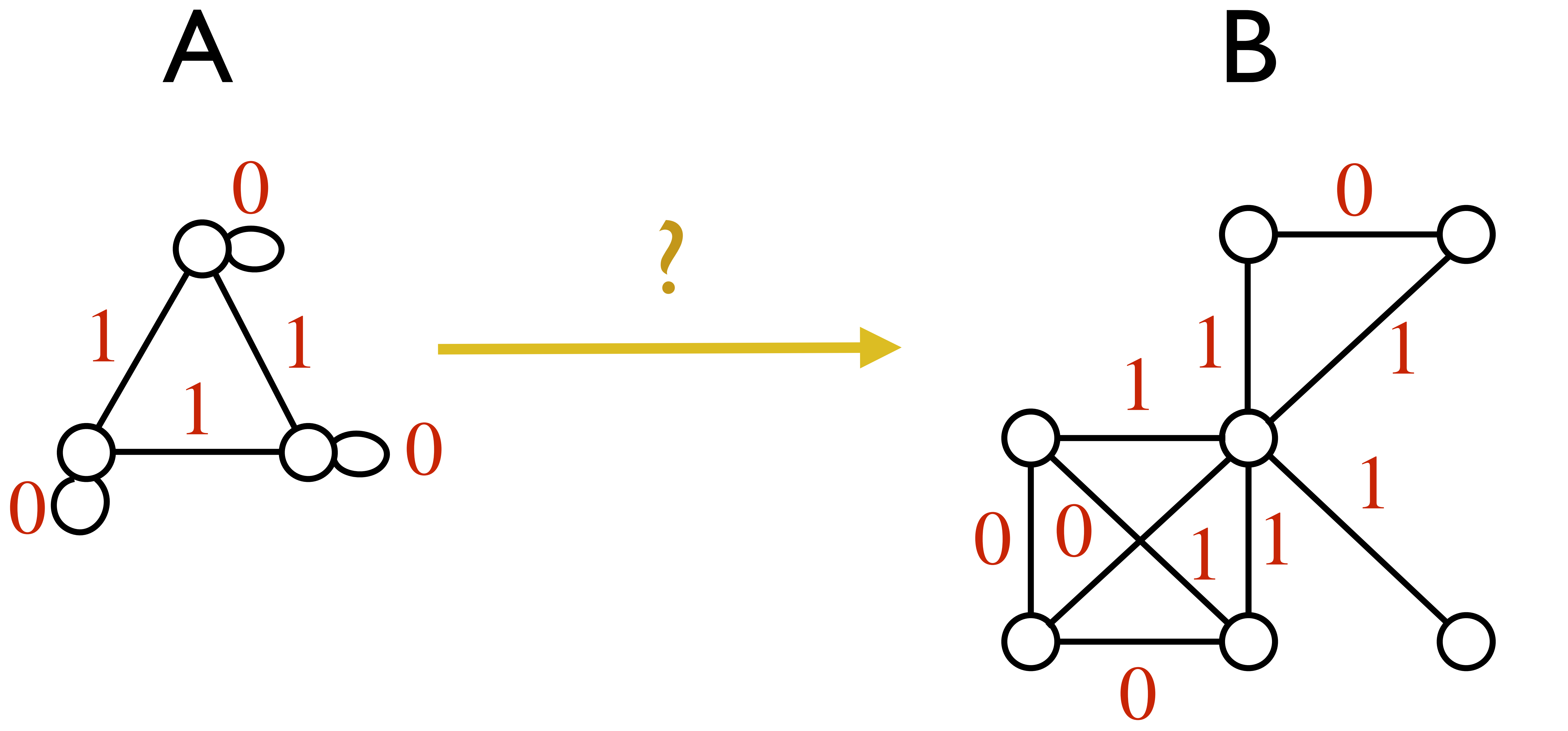
- MaxCSP(-, H) ∈ PTIME or NP-complete *[Jonsson-Krokhin JCSS'07]*
- MaxCSP(-, B) ∈ PTIME or NP-complete *[Thapper-**Ž.** JACM'16]*

MaxCSP(-, B)

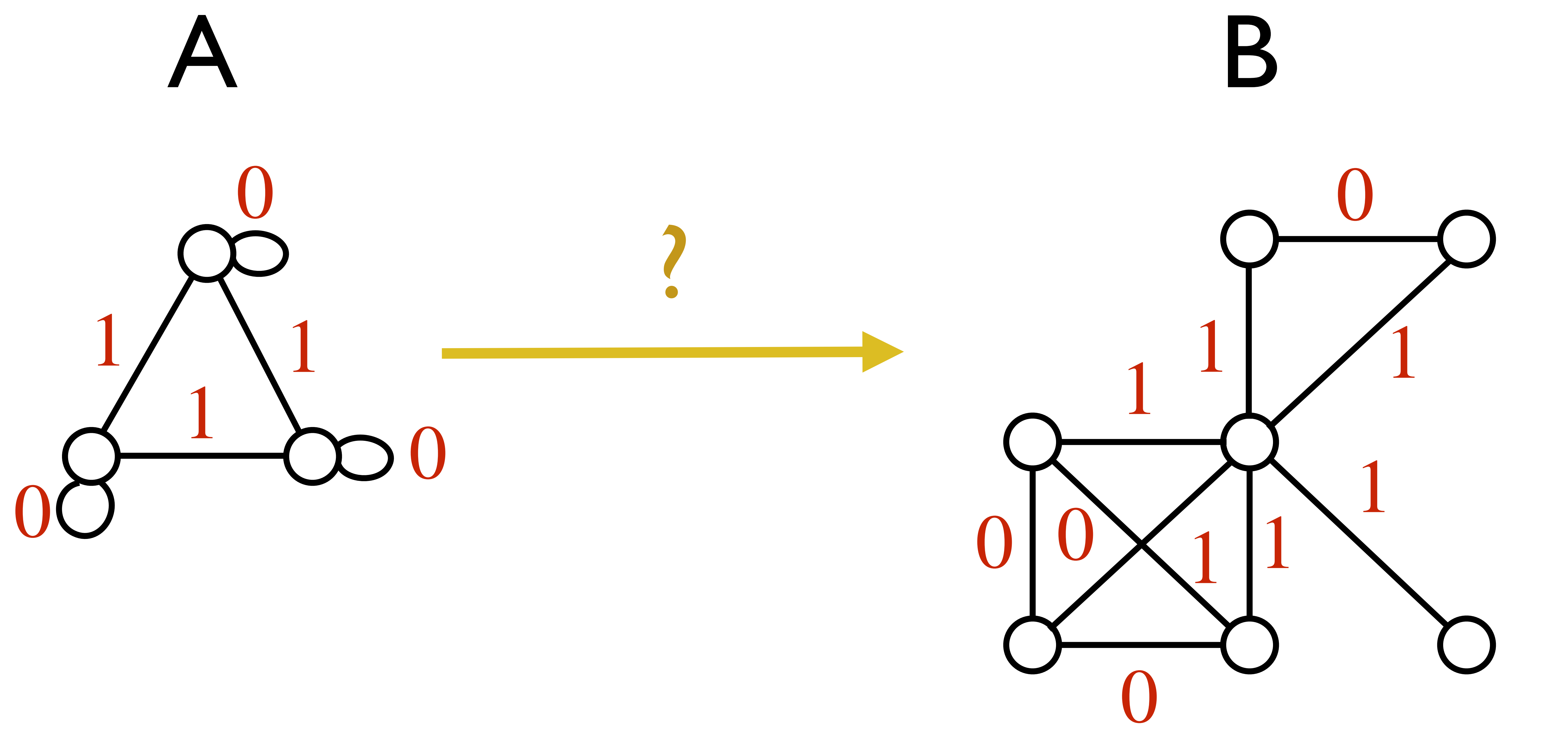


- MaxCSP(-, H) ∈ PTIME or NP-complete *[Jonsson-Krokhin JCSS'07]*
- MaxCSP(-, B) ∈ PTIME or NP-complete *[Thapper-**Ž.** JACM'16]*
- Basic SDP optimal for MaxCSP(-, B), under UGC *[Raghavendra STOC'08]*

MaxCSP($\mathscr{A}$, -)



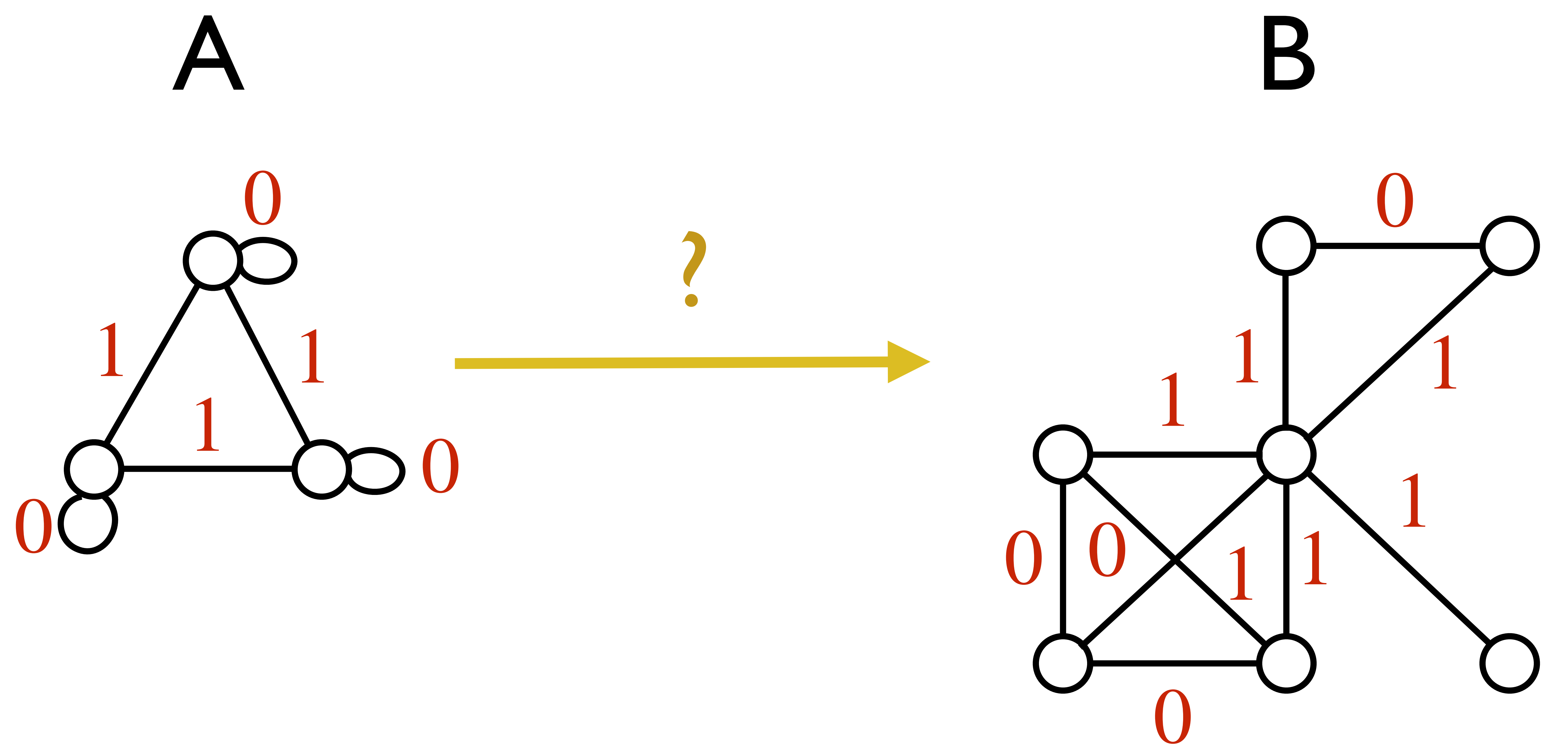
MaxCSP($\mathscr{A}$, -)



- MaxCSP($\mathscr{A}$, -) $\in$ PTIME if tw(vcore($\mathscr{A}$)) bounded, and $\notin$ PTIME otherwise

*[Carbonnel-Romero-**Ž.** SICOMP'22]*

MaxCSP($\mathscr{A}$, -)



- MaxCSP($\mathscr{A}$, -) $\in$ PTIME *[Carbonnel-Romero-**Ž.** SICOMP'22]*
  if tw(vcore($\mathscr{A}$)) bounded, and $\notin$ PTIME otherwise
- MaxCSP($\mathscr{A}_{\mathscr{G}}$, -) $\in$ APX for monotone $\mathscr{G}$ of bounded avg deg,
  Gap-ETH-hard if avg deg $\geq n^{\delta}$ *[Dinur-Manurangsi ITCS'18]*

MaxCSP($\mathscr{A}$, -)

Which $\mathscr{A}$ give rise to a PTAS for MaxCSP($\mathscr{A}$, -)?

MaxCSP($\mathscr{A}$, -)

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MaxCSP($\mathscr{A}$, -)

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$(1 \pm \varepsilon)$-approx in time $n^{f(1/\varepsilon)}$

MaxCSP($\mathscr{A}$, -)

Which $\mathscr{A}$ give rise to a PTAS for MaxCSP($\mathscr{A}$, -)?

MaxCSP($\mathscr{A}$, -)

Which $\mathscr{A}$ give rise to a PTAS for MaxCSP($\mathscr{A}$, -)?

Input: A, B with $A \in \mathscr{A}, \varepsilon > 0$

MaxCSP($\mathscr{A}$, -)

Which $\mathscr{A}$ give rise to a PTAS for MaxCSP($\mathscr{A}$, -)?

Input: A, B with $A \in \mathscr{A}, \varepsilon > 0$

Time: $(|A| + |B|)^{f(1/\varepsilon)}$

MaxCSP($\mathscr{A}$, -)

Which $\mathscr{A}$ give rise to a PTAS for MaxCSP($\mathscr{A}$, -)?

Input: A, B with $A \in \mathscr{A}, \varepsilon > 0$

Time: $(|A| + |B|)^{f(1/\varepsilon)}$

Output: $h : A \to B$ with $\mathrm{val}(h) \geq (1 - \varepsilon)\mathrm{opt}(A, B)$

MaxCSP($\mathscr{A}$, -)

Which $\mathscr{A}$ give rise to a PTAS for MaxCSP($\mathscr{A}$, -)?

Input: A, B with $A \in \mathscr{A}, \varepsilon > 0$

Time: $(|A| + |B|)^{f(1/\varepsilon)}$

Output: ~~$h : A \to B$ with $\mathrm{val}(h) \geq (1-\varepsilon)\mathrm{opt}(A, B)$~~

v with $v \geq (1-\varepsilon)\mathrm{opt}(A, B)$

Tw-Pliability

Tw-Pliability

"uniformly close to structures of bounded treewidth"

Tw-Pliability

"uniformly close to structures of bounded treewidth"

$\mathscr{A}$ treewidth-pliable if $\forall \varepsilon > 0 \, \exists k \in \mathbb{N} \, \forall A \in \mathscr{A} \, \exists A'$

Tw-Pliability

"uniformly close to structures of bounded treewidth"

$\mathscr{A}$ treewidth-pliable if $\forall \varepsilon > 0 \, \exists k \in \mathbb{N} \, \forall A \in \mathscr{A} \, \exists A'$

- $\mathrm{tw}(A') \leq k$
- $\mathrm{opt}(A', B) \geq (1 - \varepsilon)\mathrm{opt}(A, B)$ for all B

Tw-Pliability

"uniformly close to structures of bounded treewidth"

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- $\mathrm{opt}(A', B) \geq (1 - \varepsilon)\mathrm{opt}(A, B)$ for all B

$(U; R_1, \ldots, R_k)$

Tw-Pliability

"uniformly close to structures of bounded treewidth"

$\mathscr{A}$ treewidth-pliable if $\forall \varepsilon > 0 \, \exists k \in \mathbb{N} \, \forall A \in \mathscr{A} \, \exists A'$

- $\text{tw}(A') \leq k$
- $\text{opt}(A', B) \geq (1 - \varepsilon)\text{opt}(A, B)$ for all B

$$(U; R_1, \ldots, R_k) \xrightarrow{\text{Gaifman}} (U; \{\{uv\} \mid \exists i \, \exists \mathbf{t} \in R_i \text{ with } u, v \in R_i\})$$

Tw-Pliability $\Rightarrow$ PTAS

Thm: Let $\mathscr{A}$ be a set of $\mathbb{Q}_{\geq 0}$-valued structures that is treewidth-pliable. Then, MaxCSP($\mathscr{A}$, -) admits a PTAS.

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$$d_{opt}(A, A') = \inf_{\varepsilon}[\forall B \mid \text{MaxCSP}(A, B) = (1 \pm \varepsilon)\text{MaxCSP}(A', B)]$$

Sparse Structures

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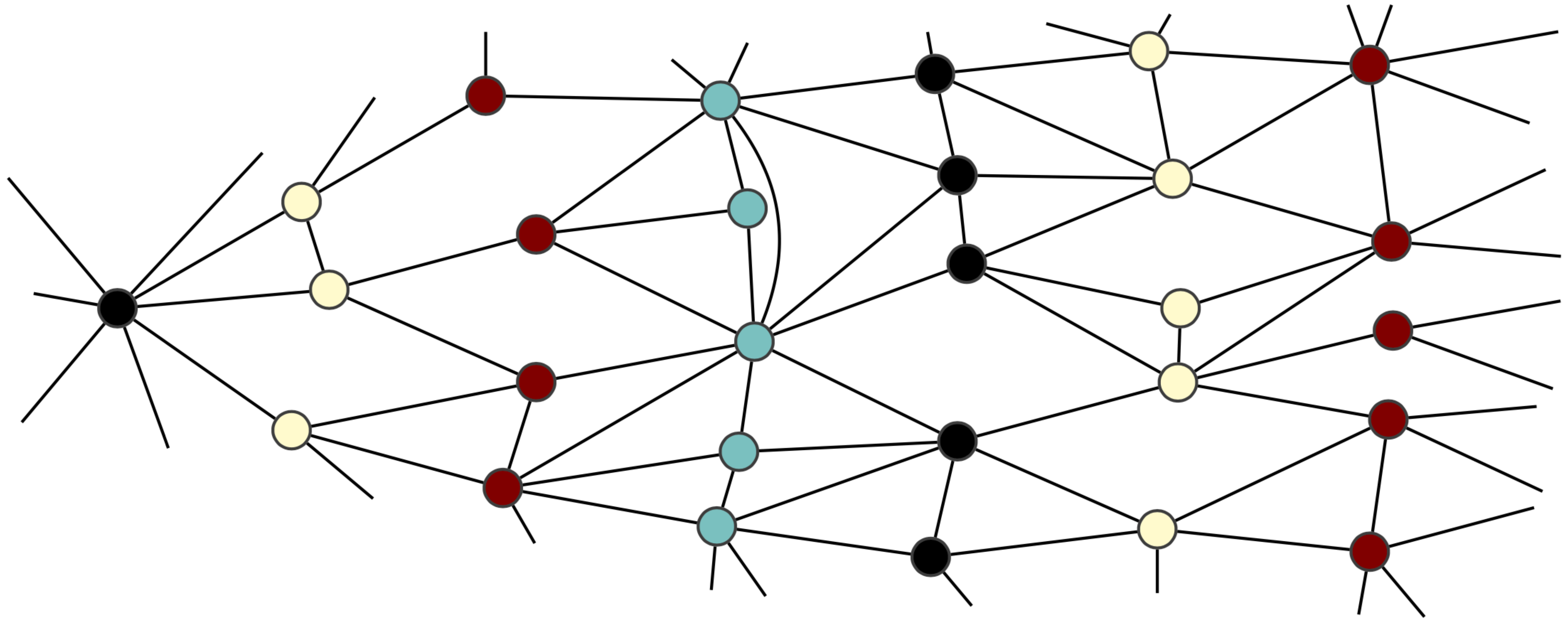

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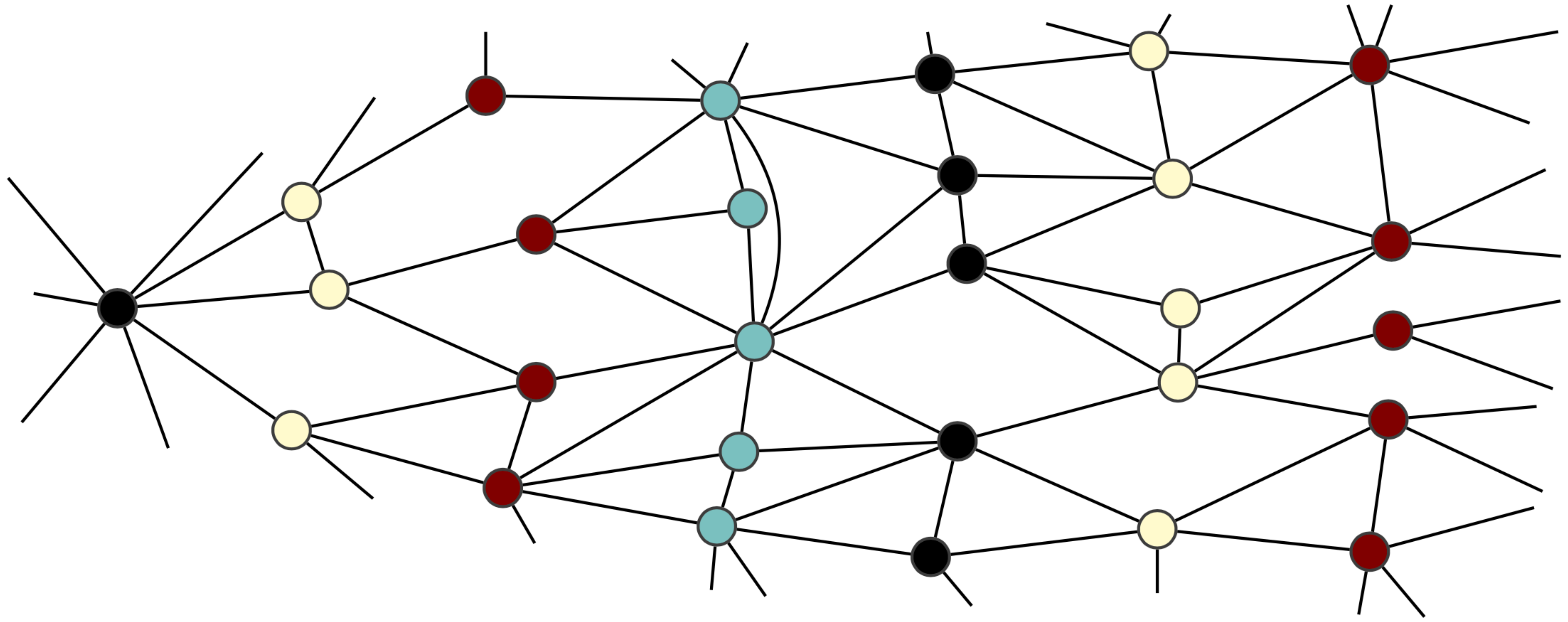

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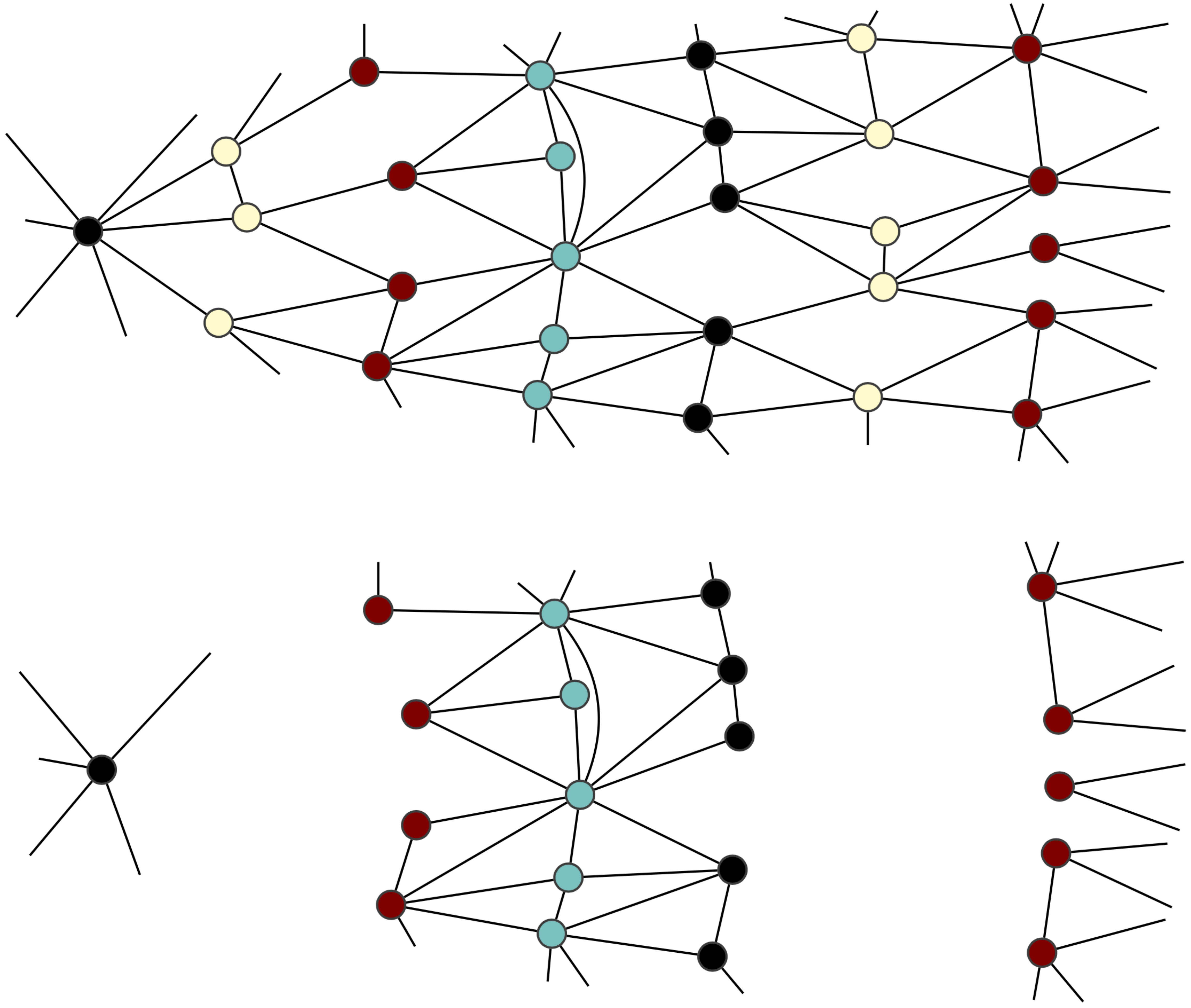

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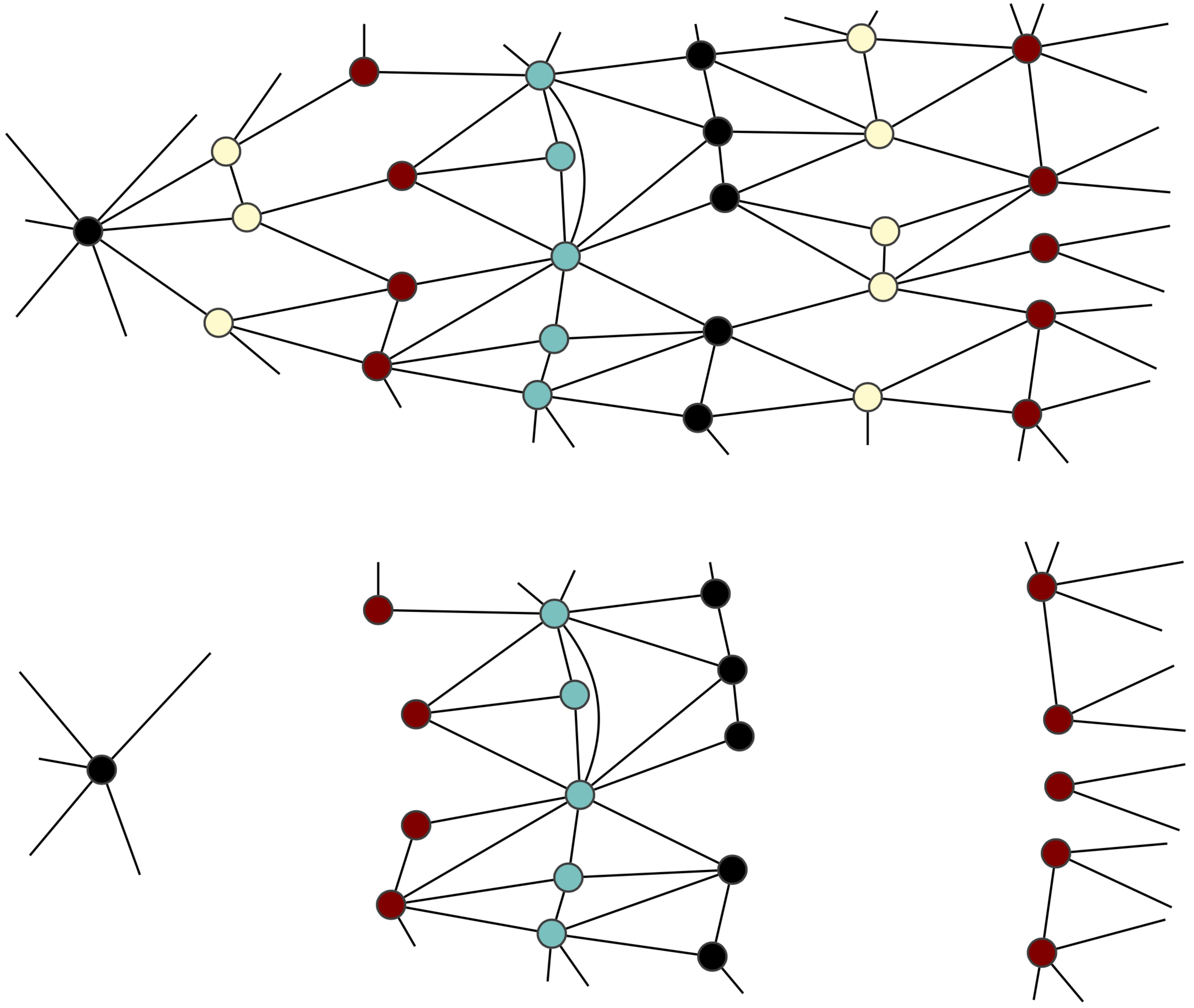

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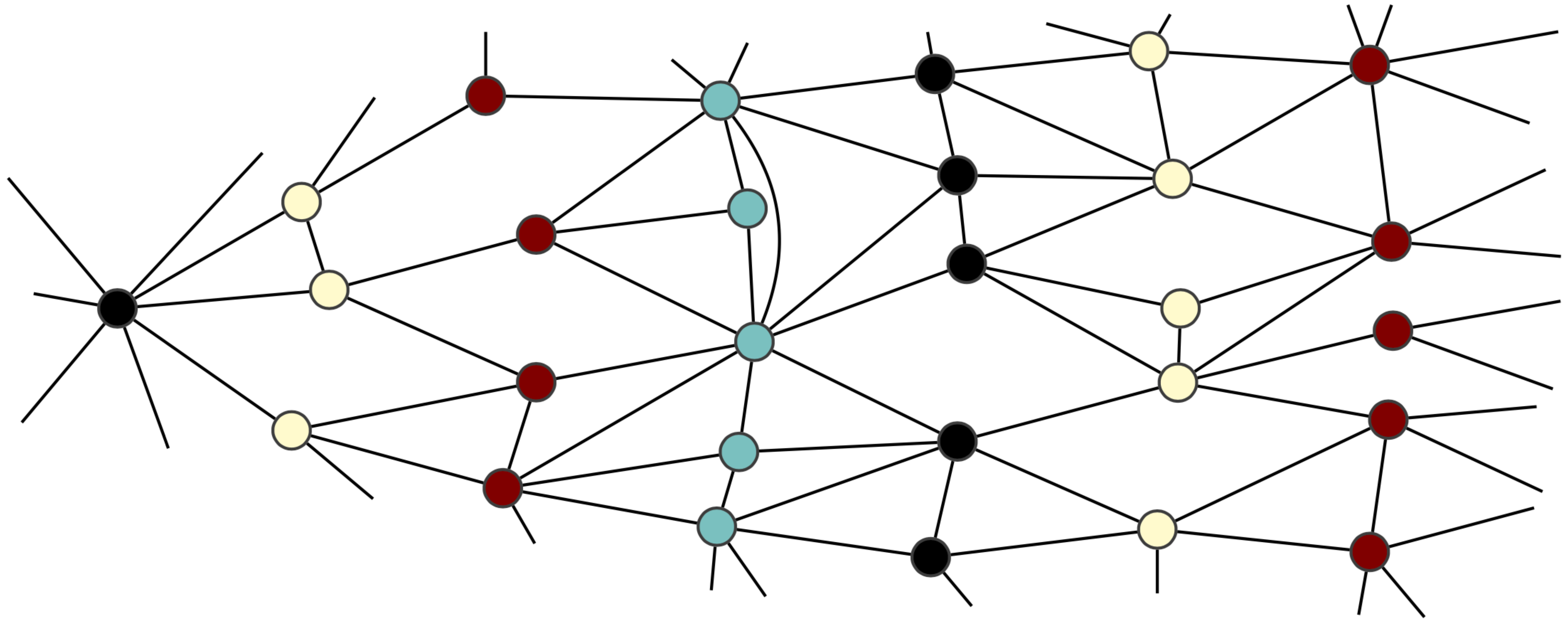
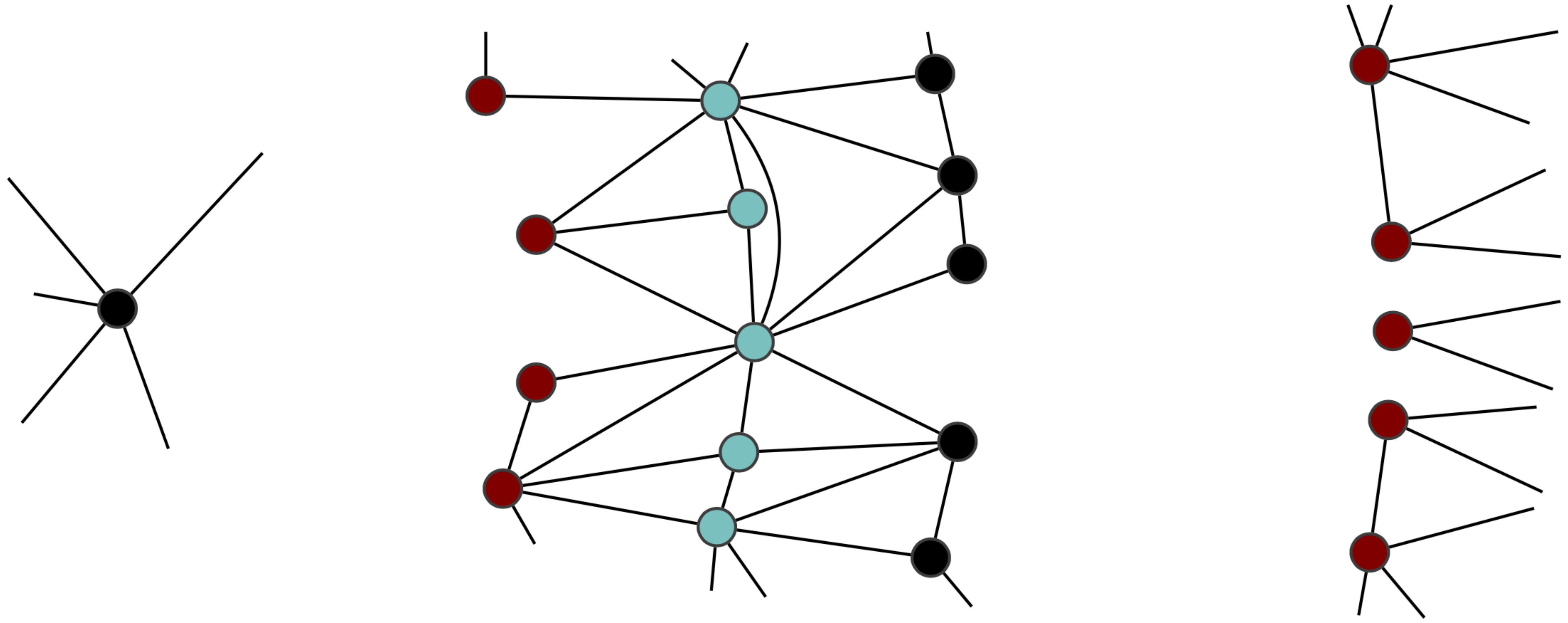


- minor-free classes

[Grohe Comb'03, Demaine et al. FOCS'05]



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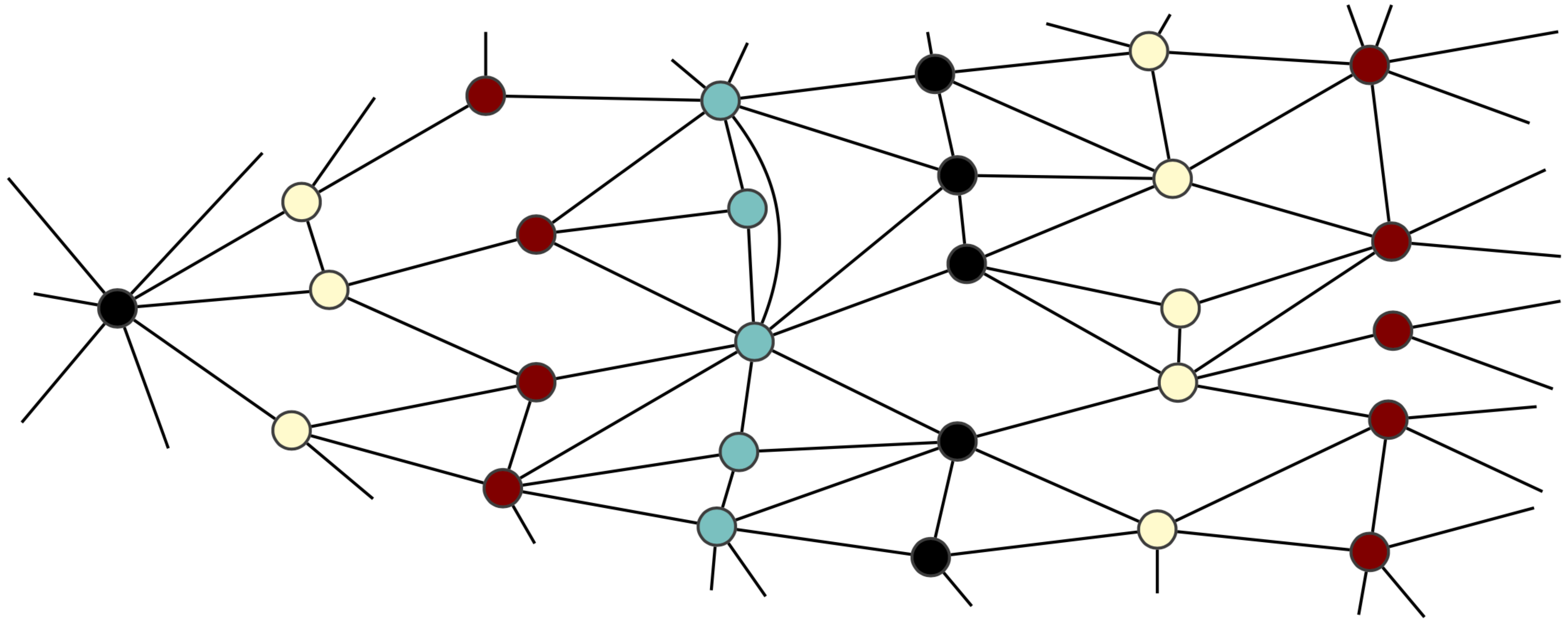

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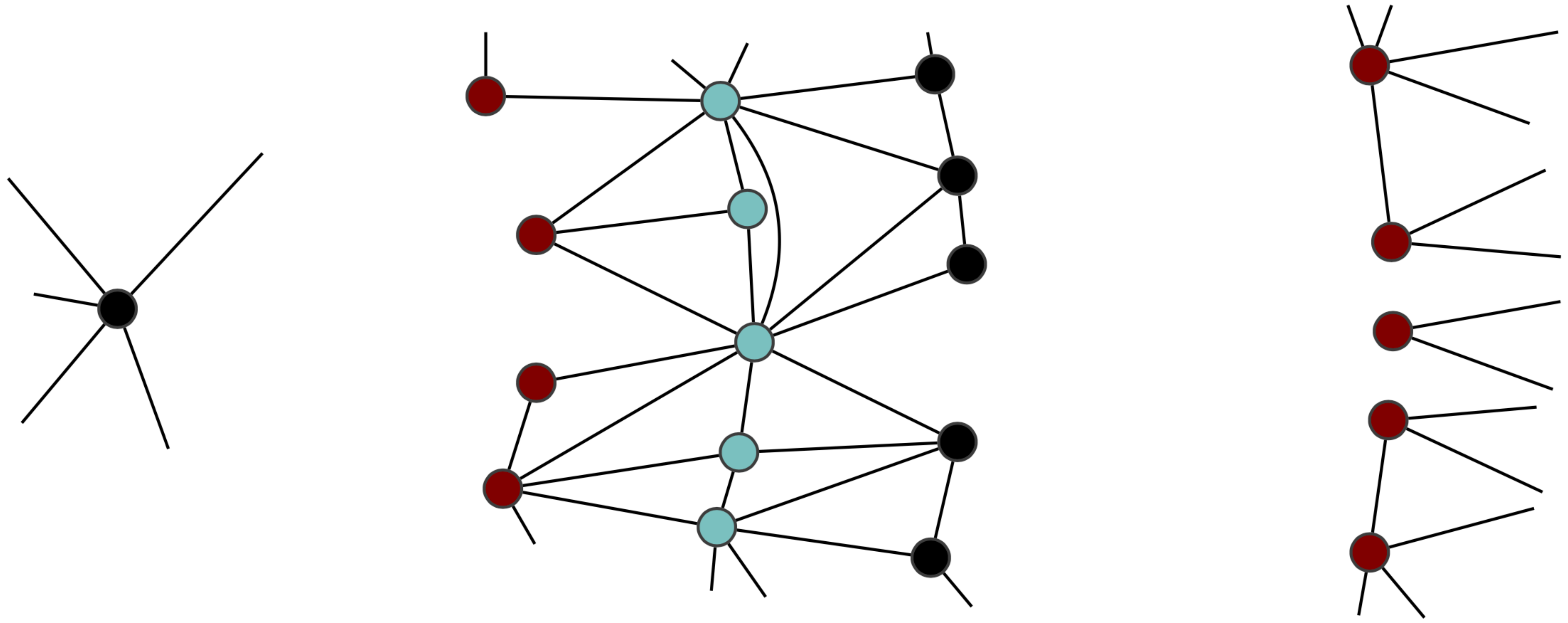

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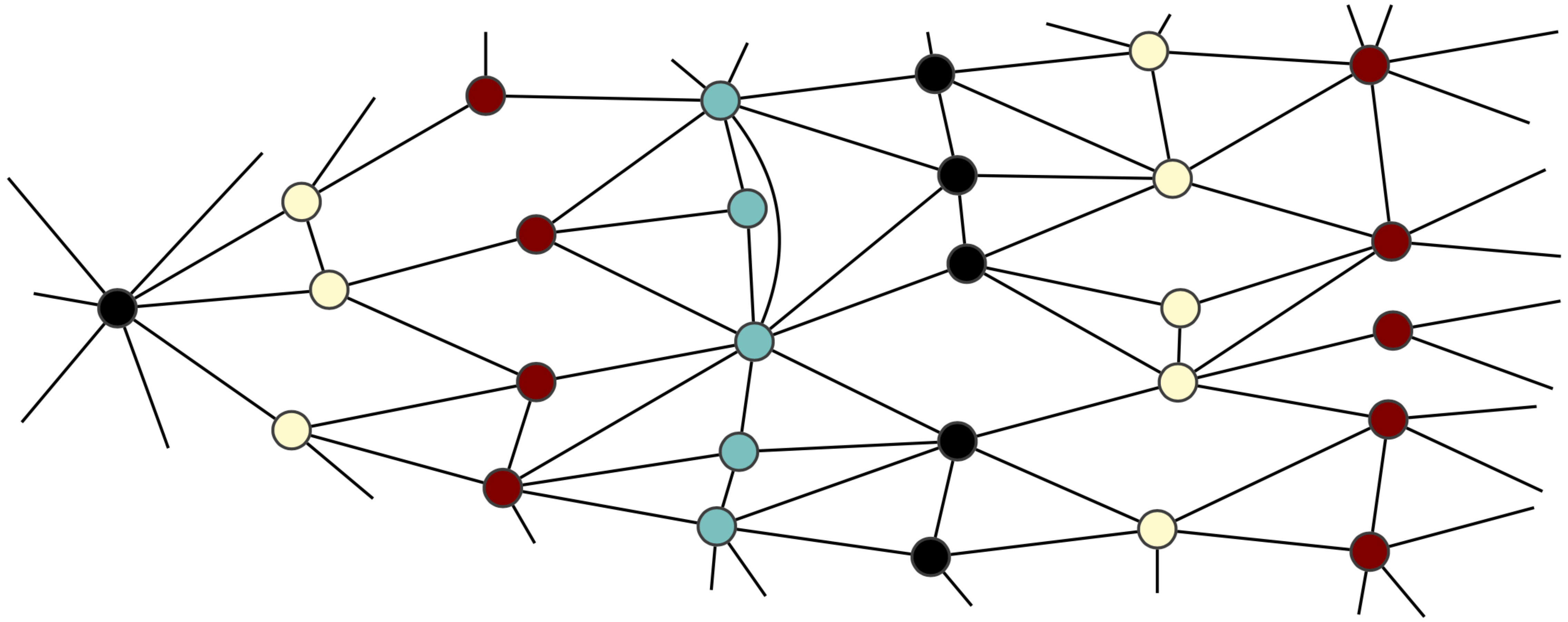

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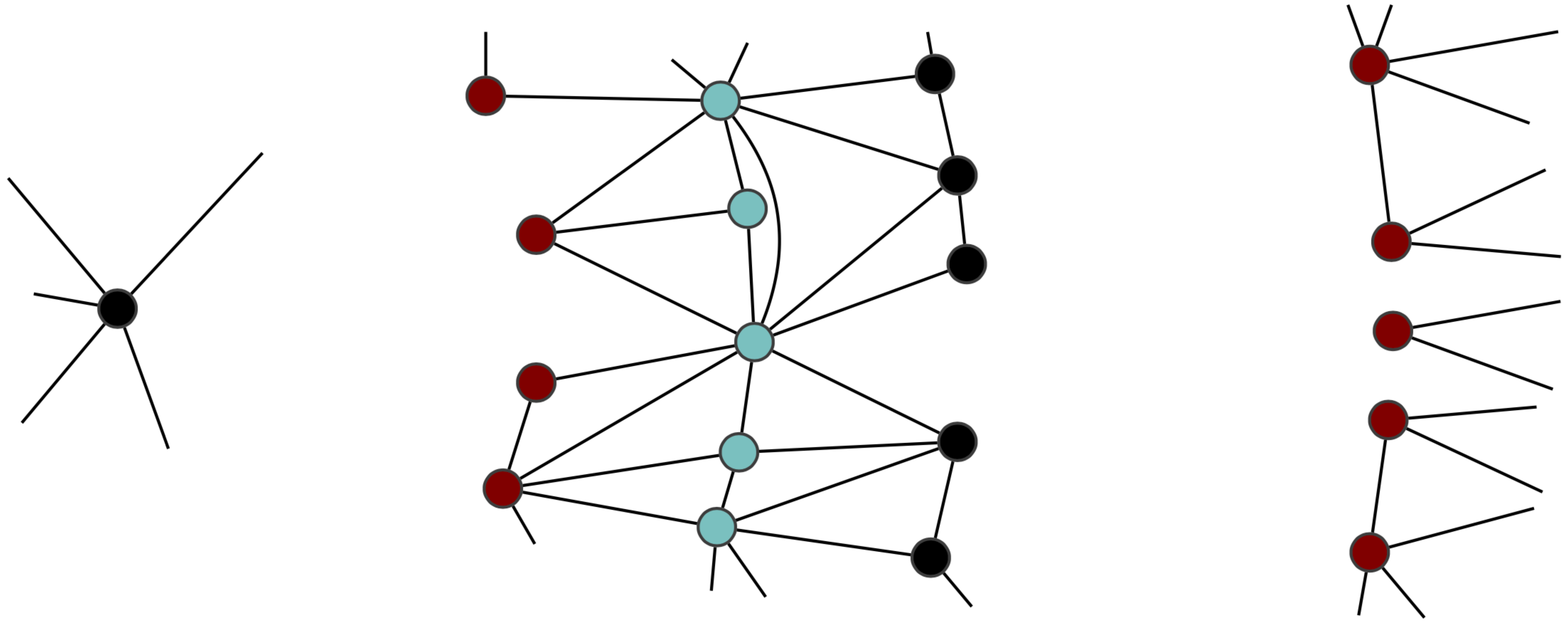

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$\mathcal{G}$ fr-tw-fragile if $\forall \varepsilon > 0\, \exists k \in \mathbb{N}\, \forall G \in \mathcal{G}\, \exists$ distribution on $\{X \subseteq V(G) \mid \mathrm{tw}(G - X) \leq k\}$ with, $\forall v, \mathrm{Prob}[v \in X] \leq \varepsilon$

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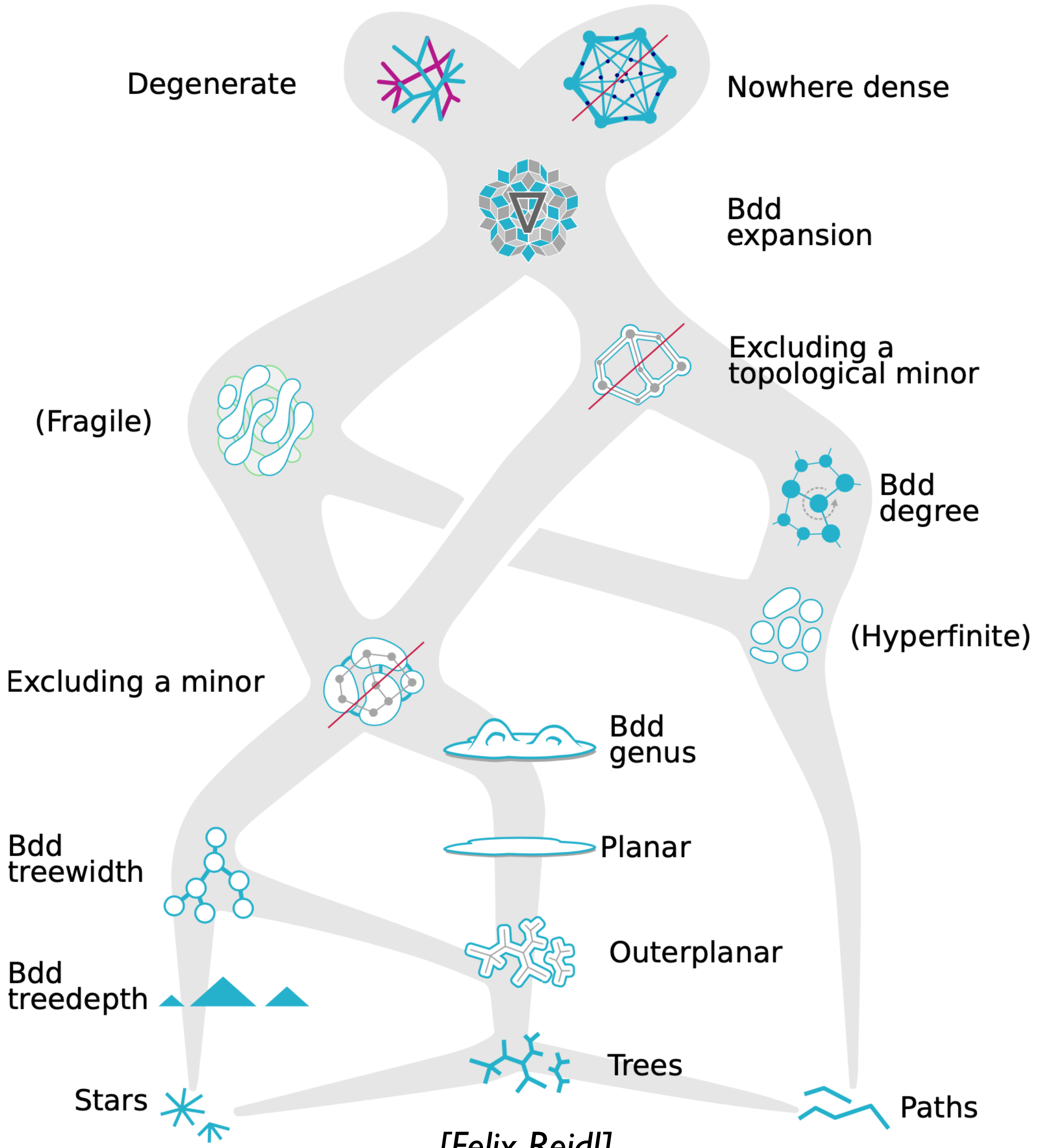
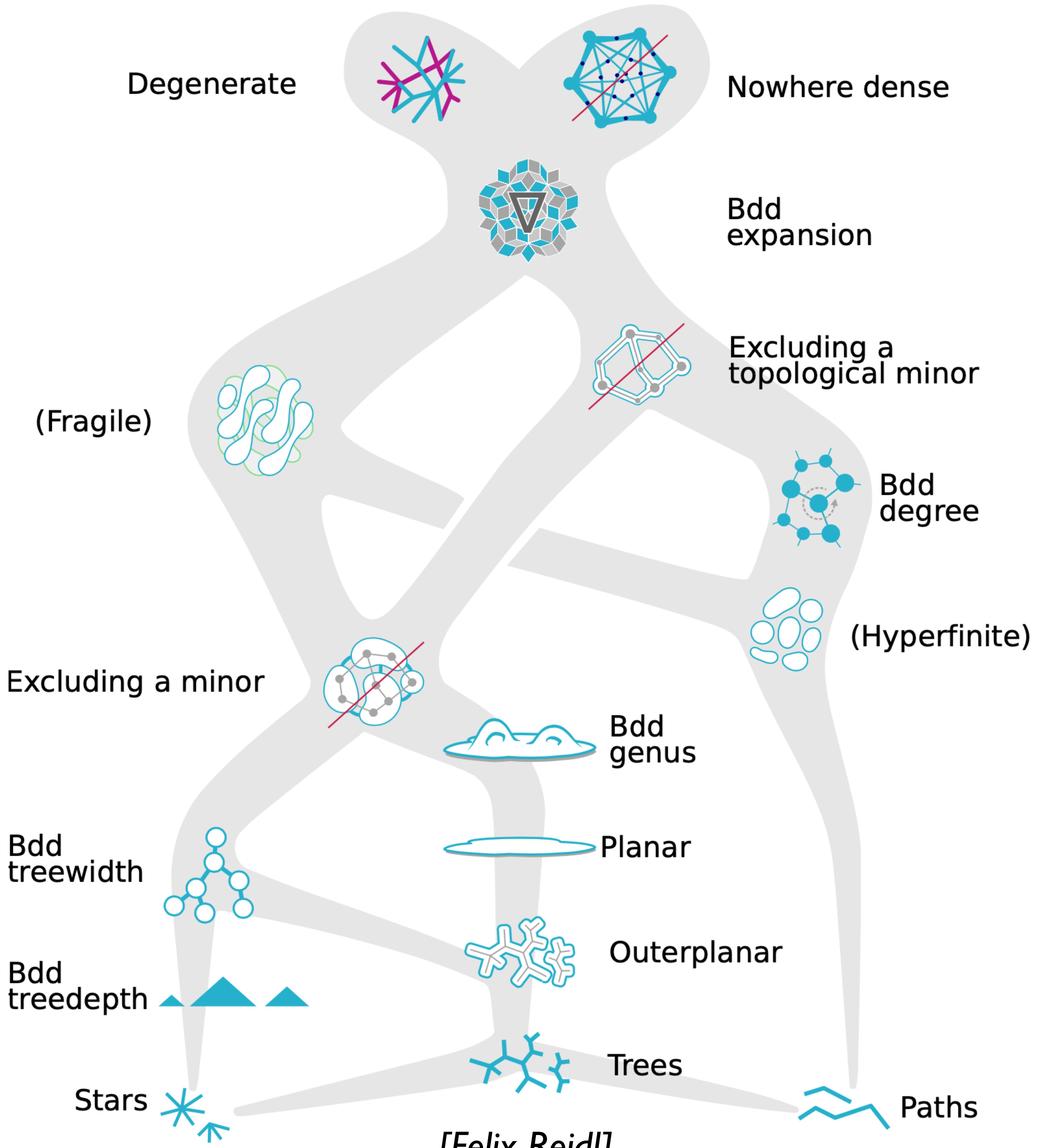

Degenerate
Nowhere dense
Bdd expansion
Excluding a topological minor
(Fragile)
Bdd degree
(Hyperfinite)
Excluding a minor
Bdd genus
Planar
Bdd treewidth
Outerplanar
Bdd treedepth
Trees
Stars
Paths

[Felix Reidl]

Conj: MaxCSP($\mathscr{A}_{\mathscr{G}}$,-) admits a PTAS **iff** $\mathscr{G}$ is fr-tw-fr.



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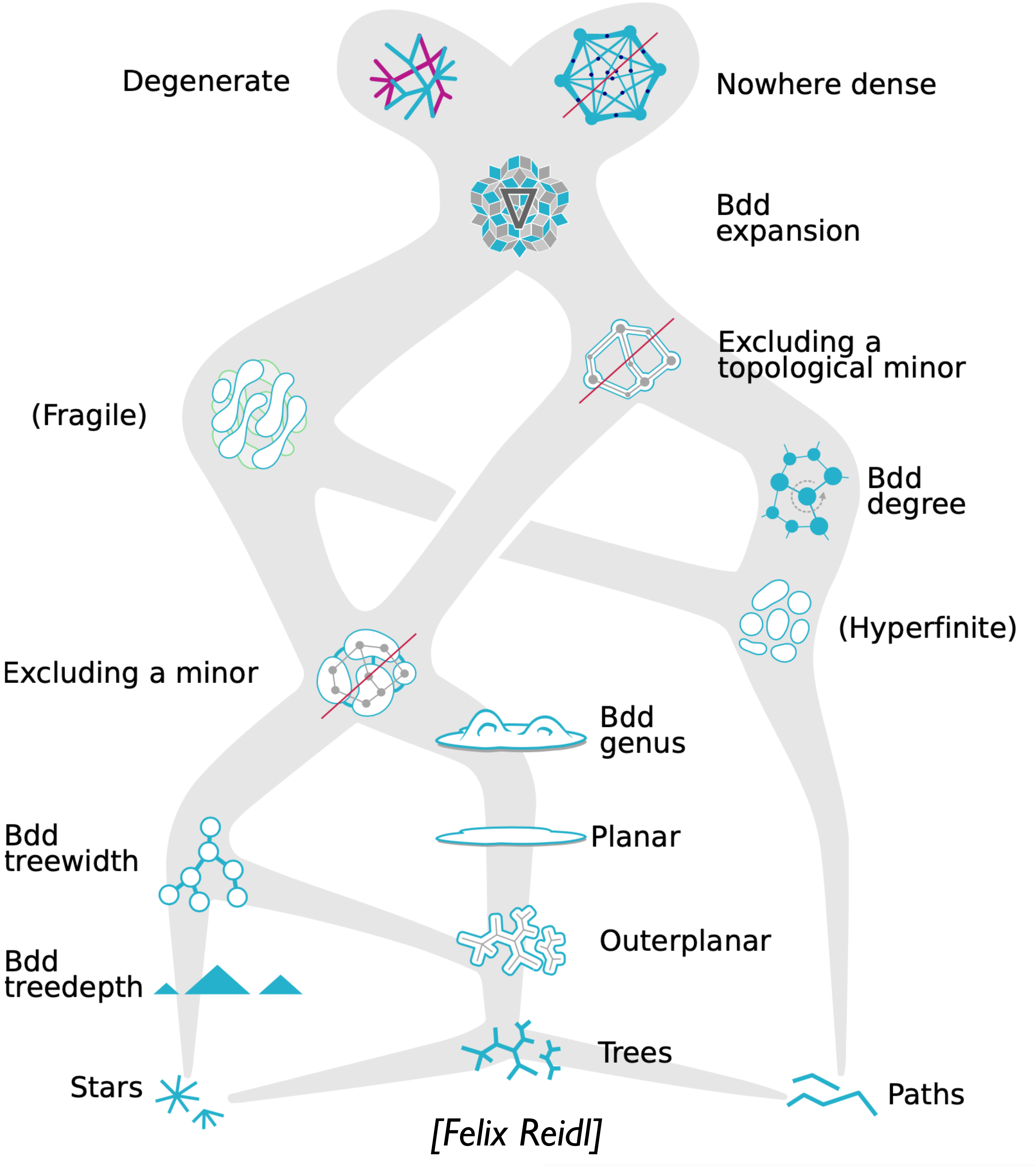

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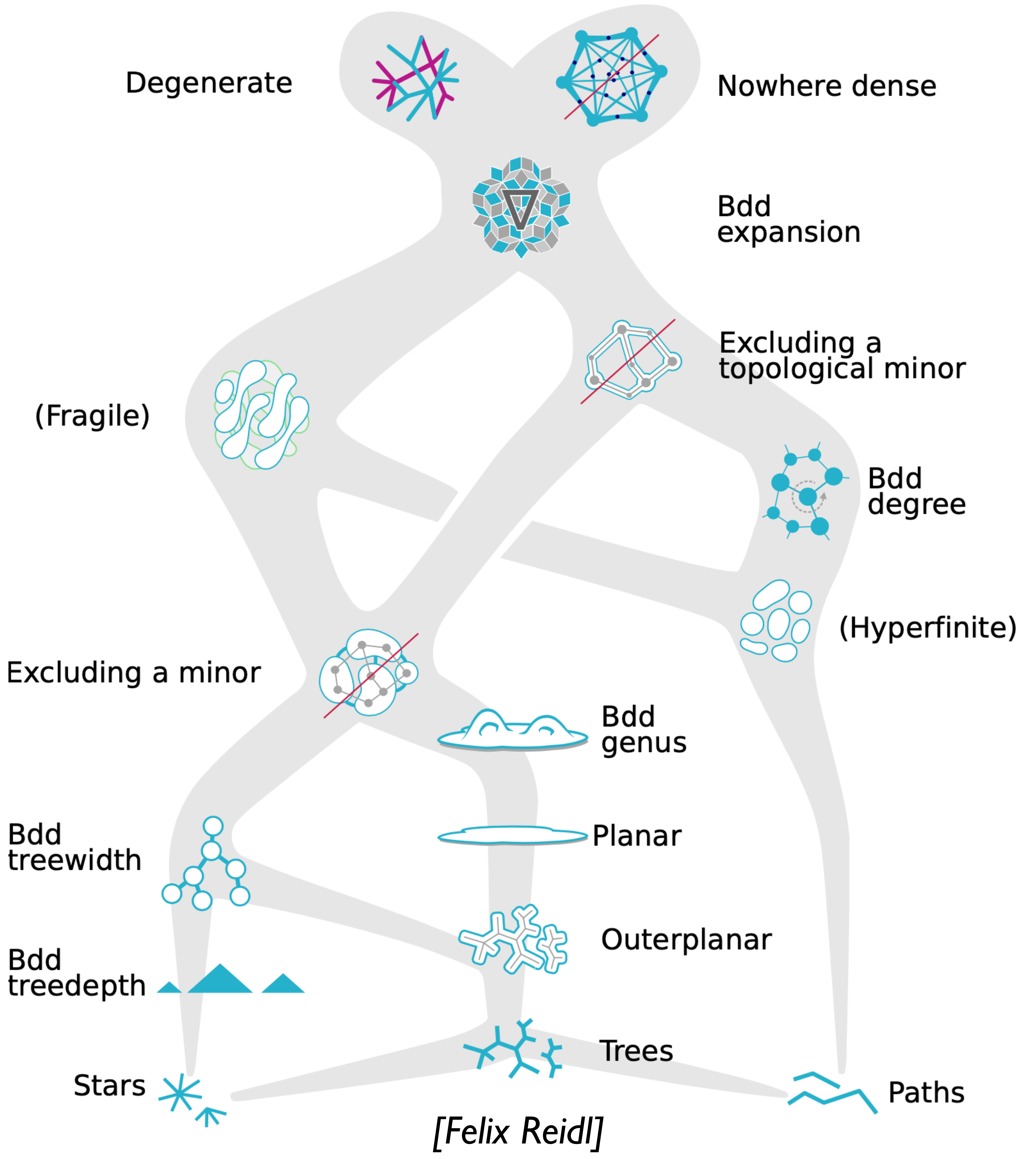

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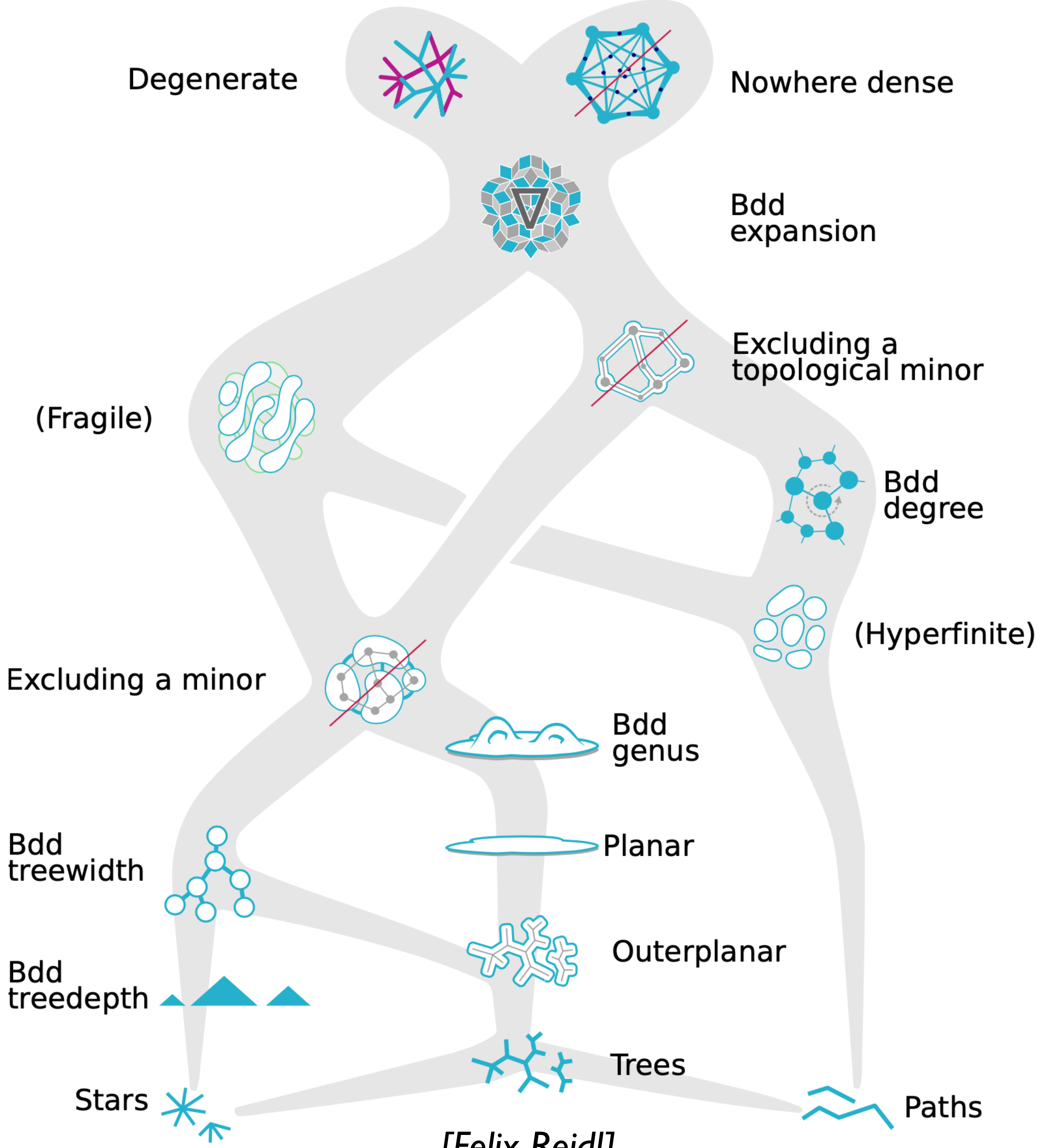

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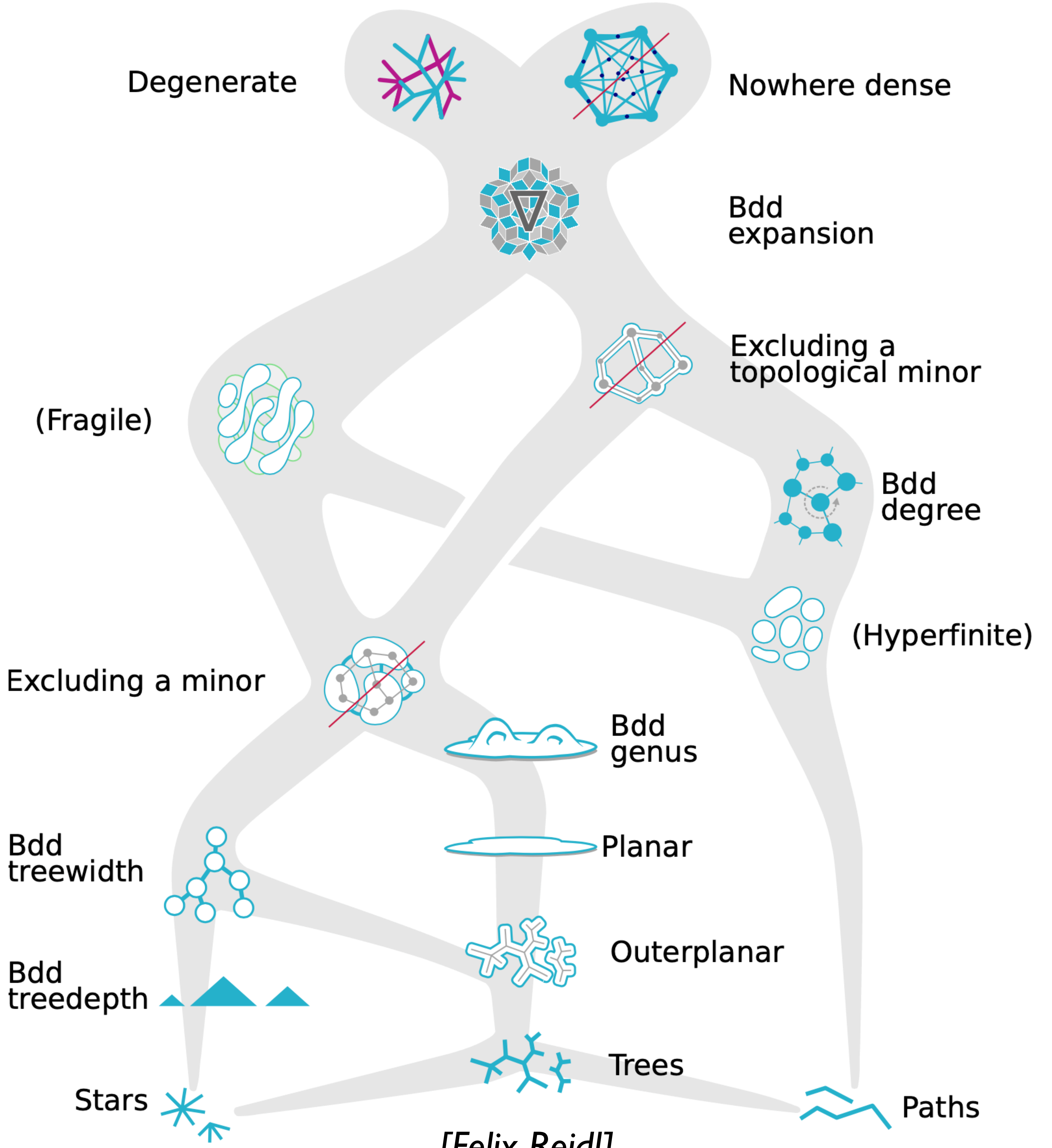

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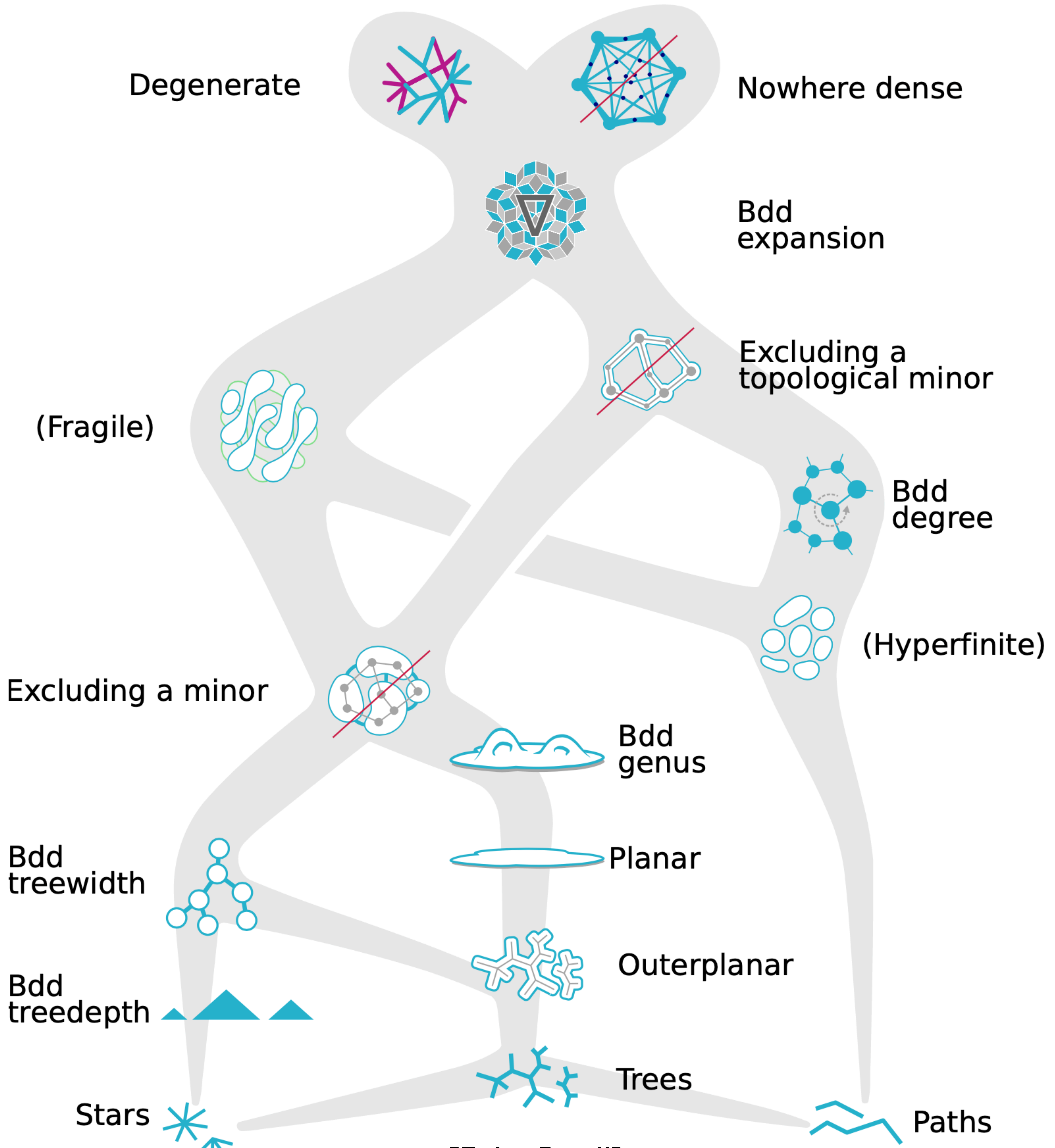

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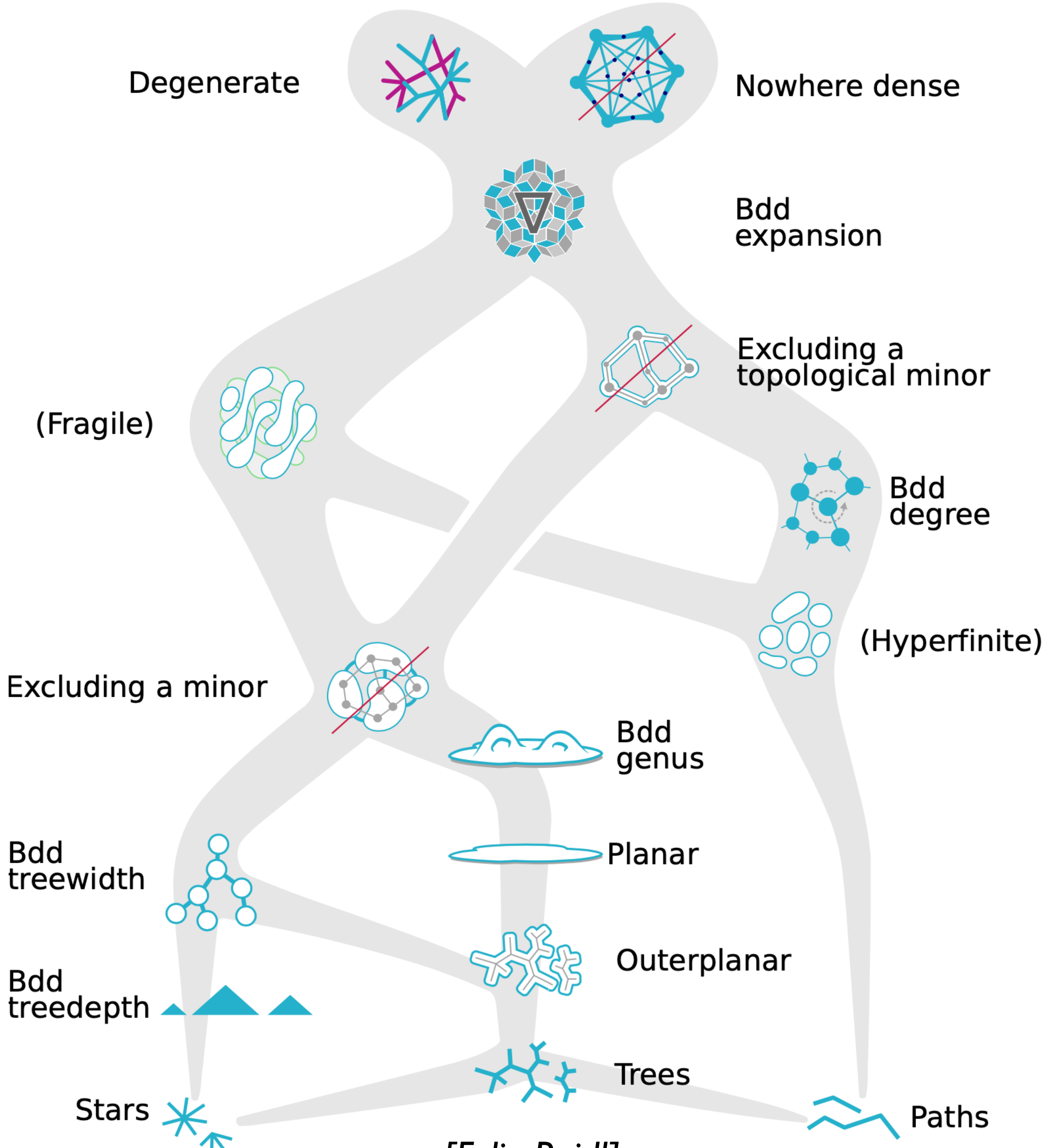

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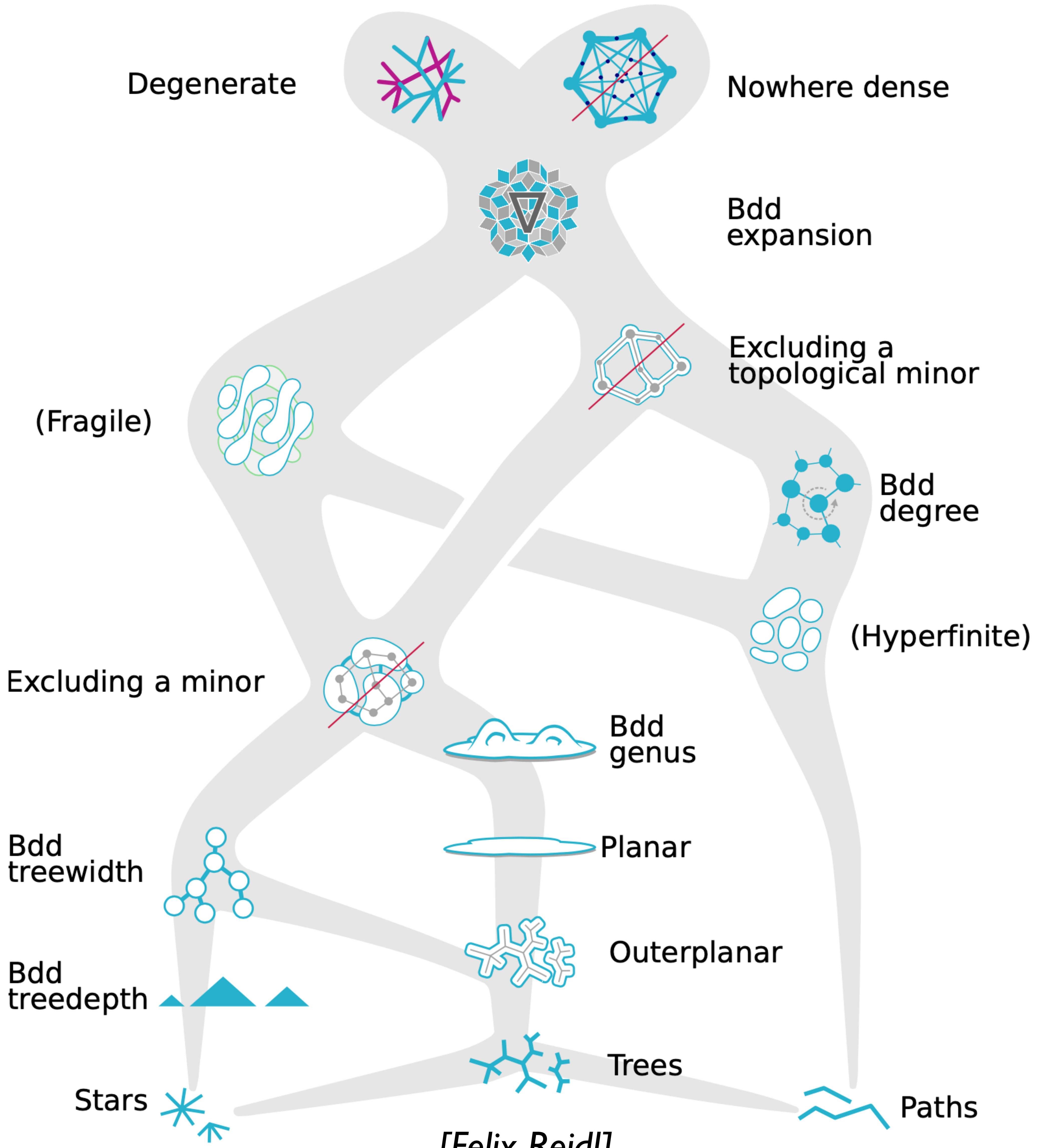

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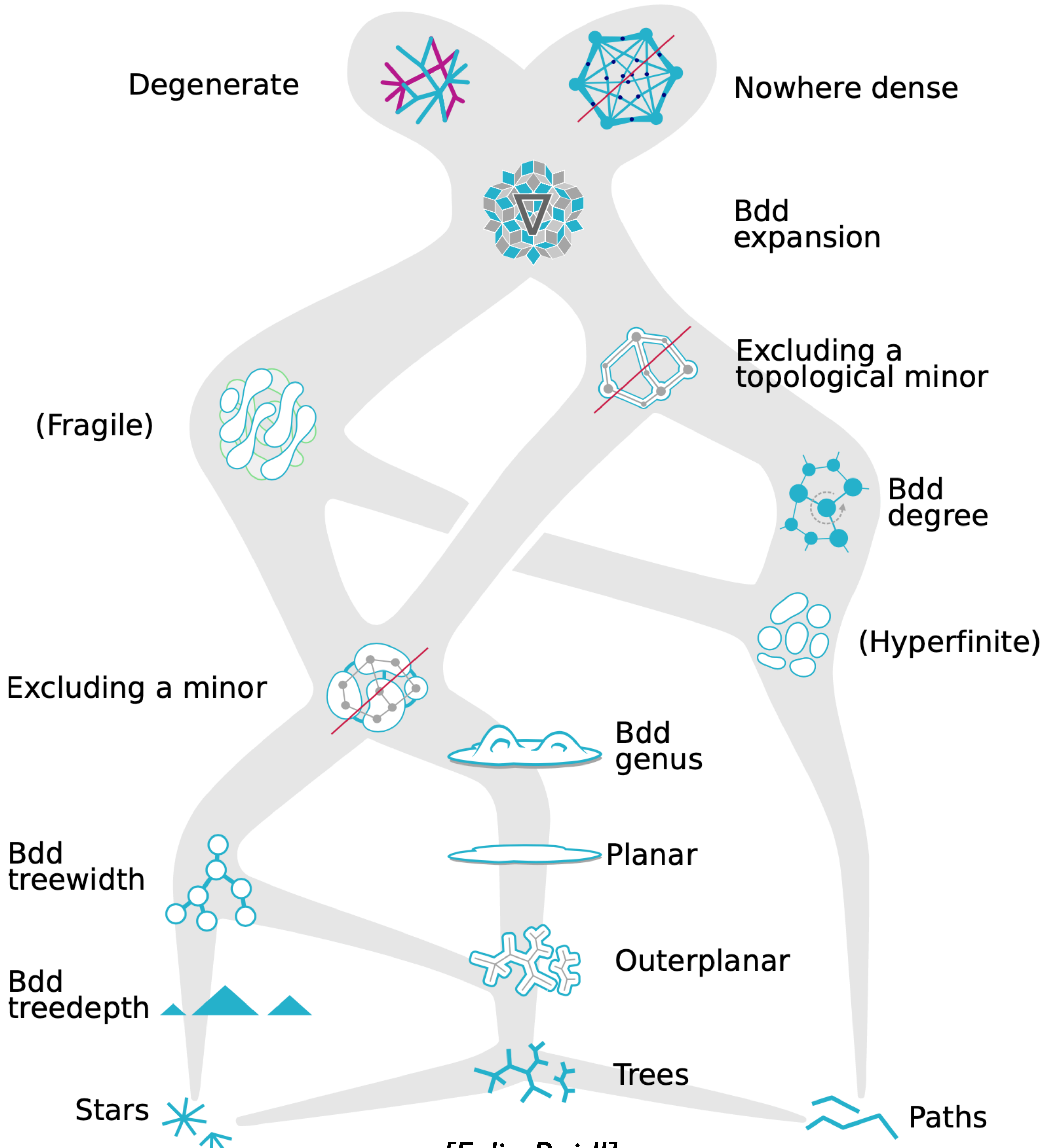

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Balázs Mezei

Miguel Romero

PUC

Marcin Wrochna

Warsaw

- Pliability and approximating MaxCSPs *[RWŽ]*
- PTAS for general sparse **general-valued** CSPs *[MWŽ]*

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$$\mathscr{G}' = \{G \cup P_n \mid G \in \mathscr{G}, n = \mathrm{mis}(G)\}$$

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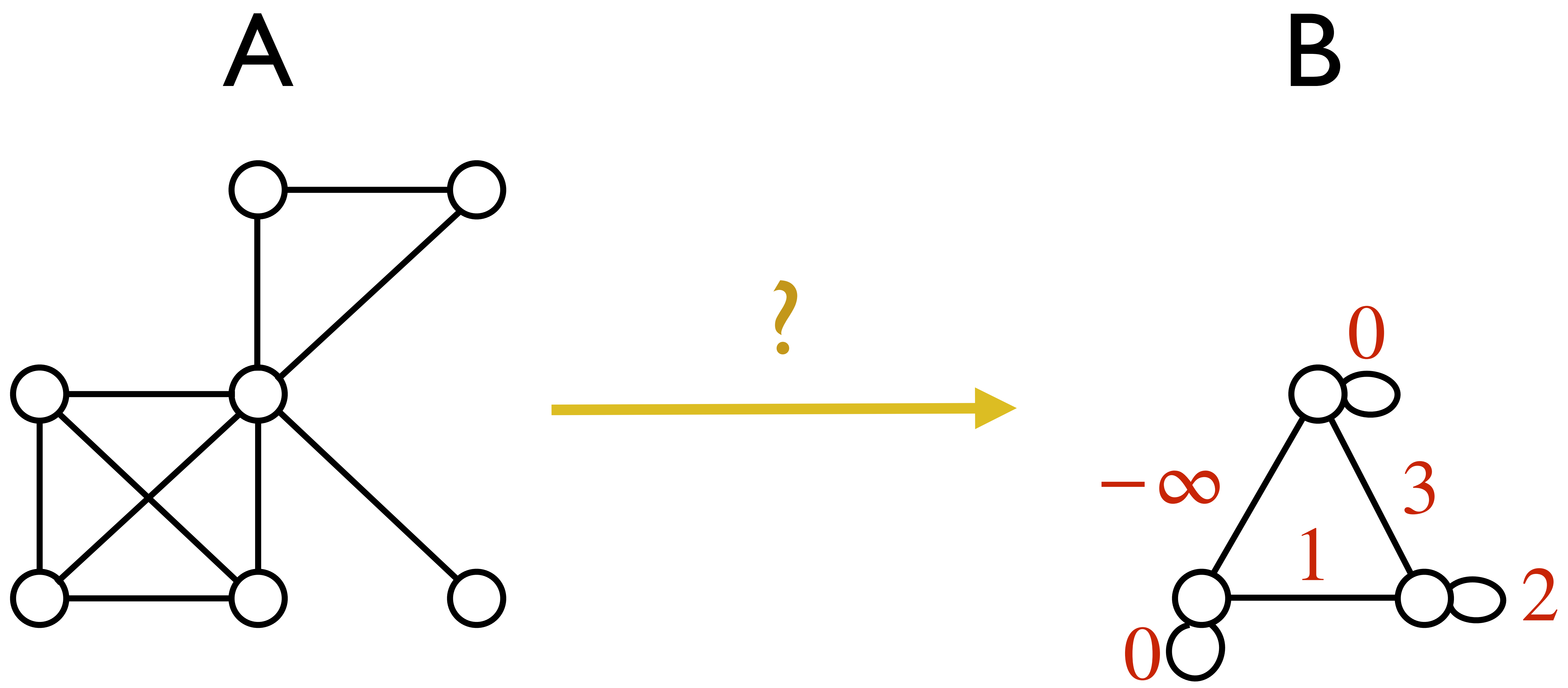

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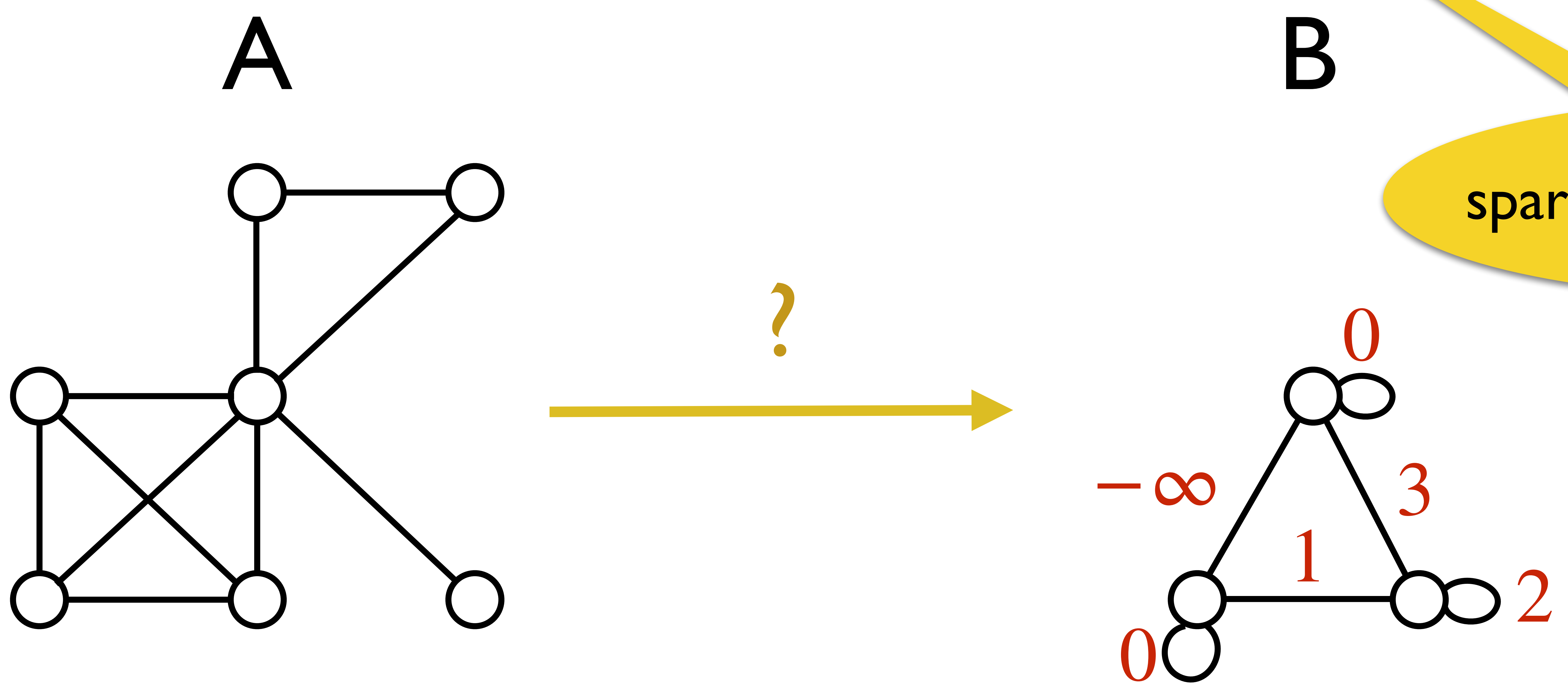

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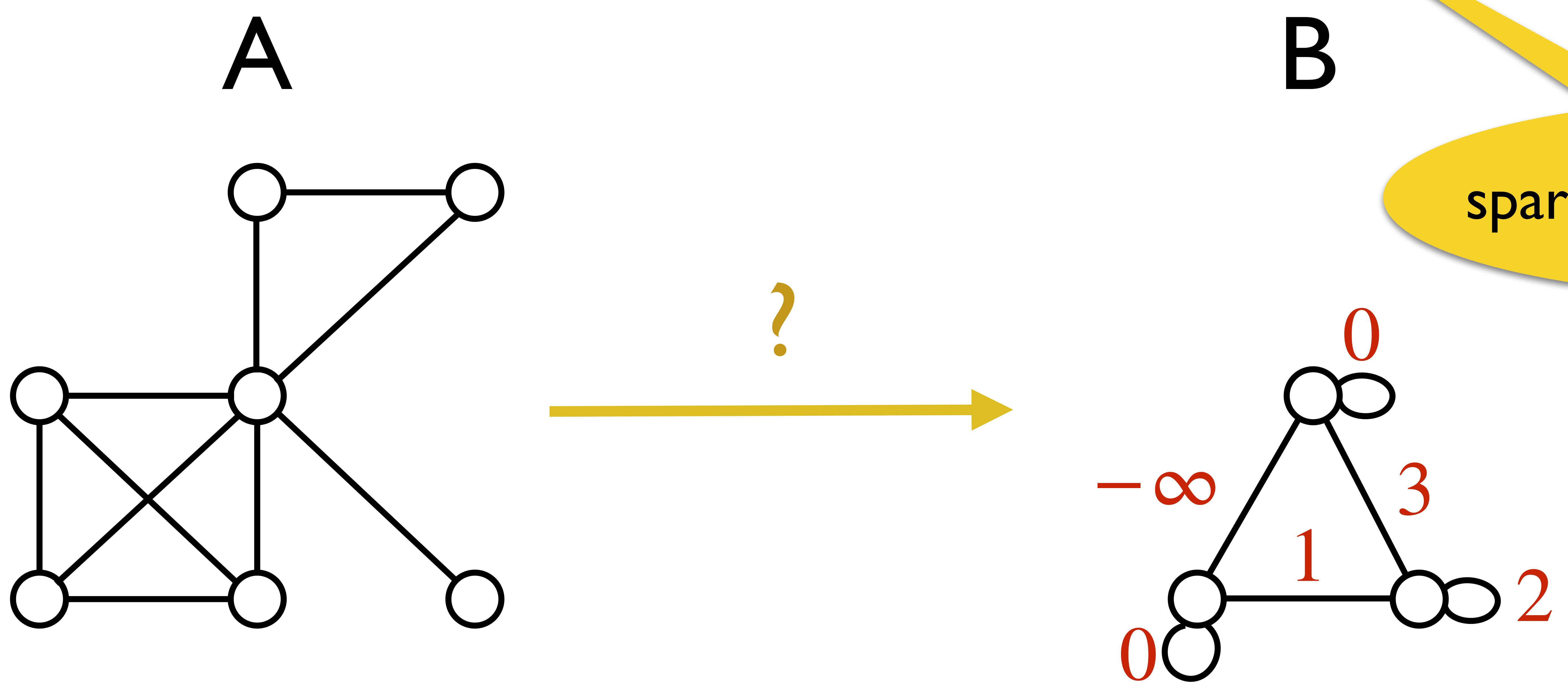

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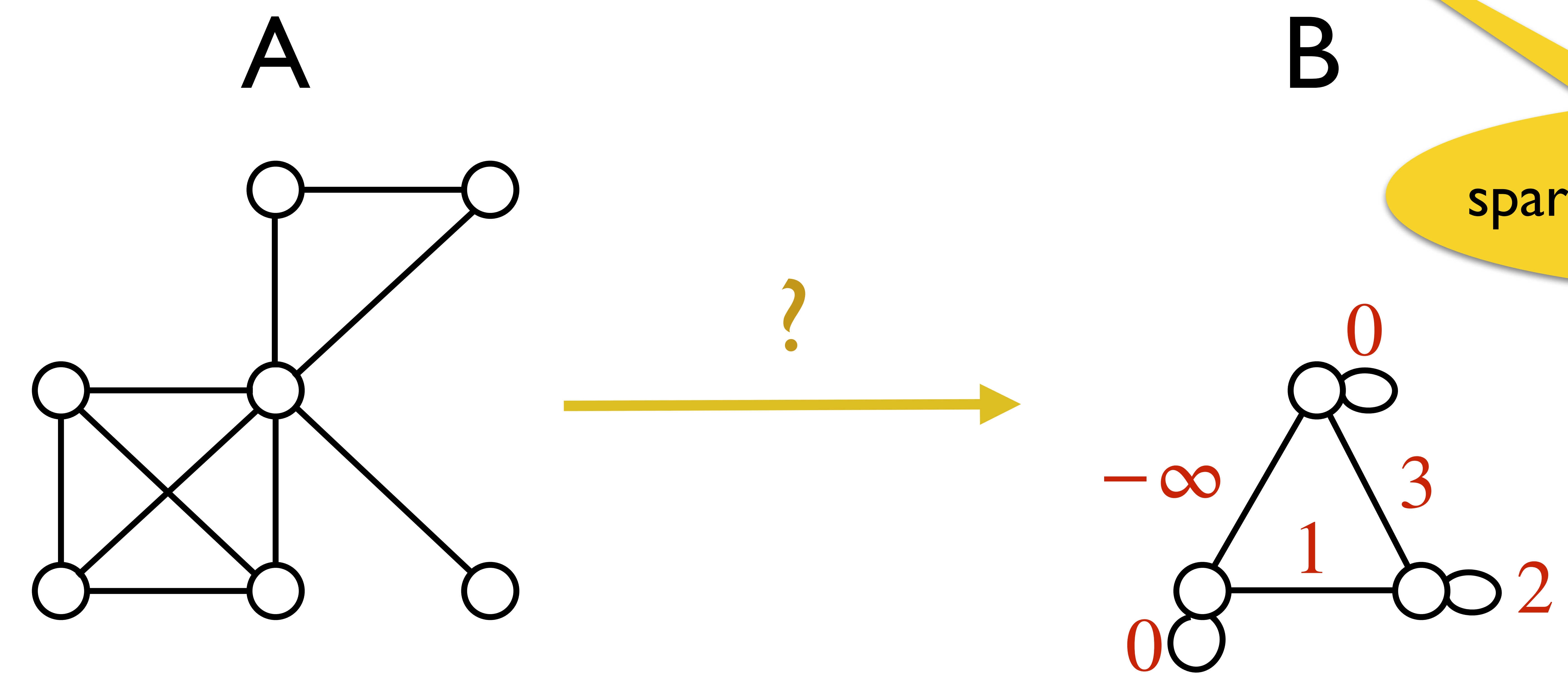

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- CSP($\mathscr{A}_{\mathscr{G}}$, -) $\in$ PTIME iff tw($\mathscr{G}$) bounded *[Grohe-Schwentick-Segoufin STOC'01]*

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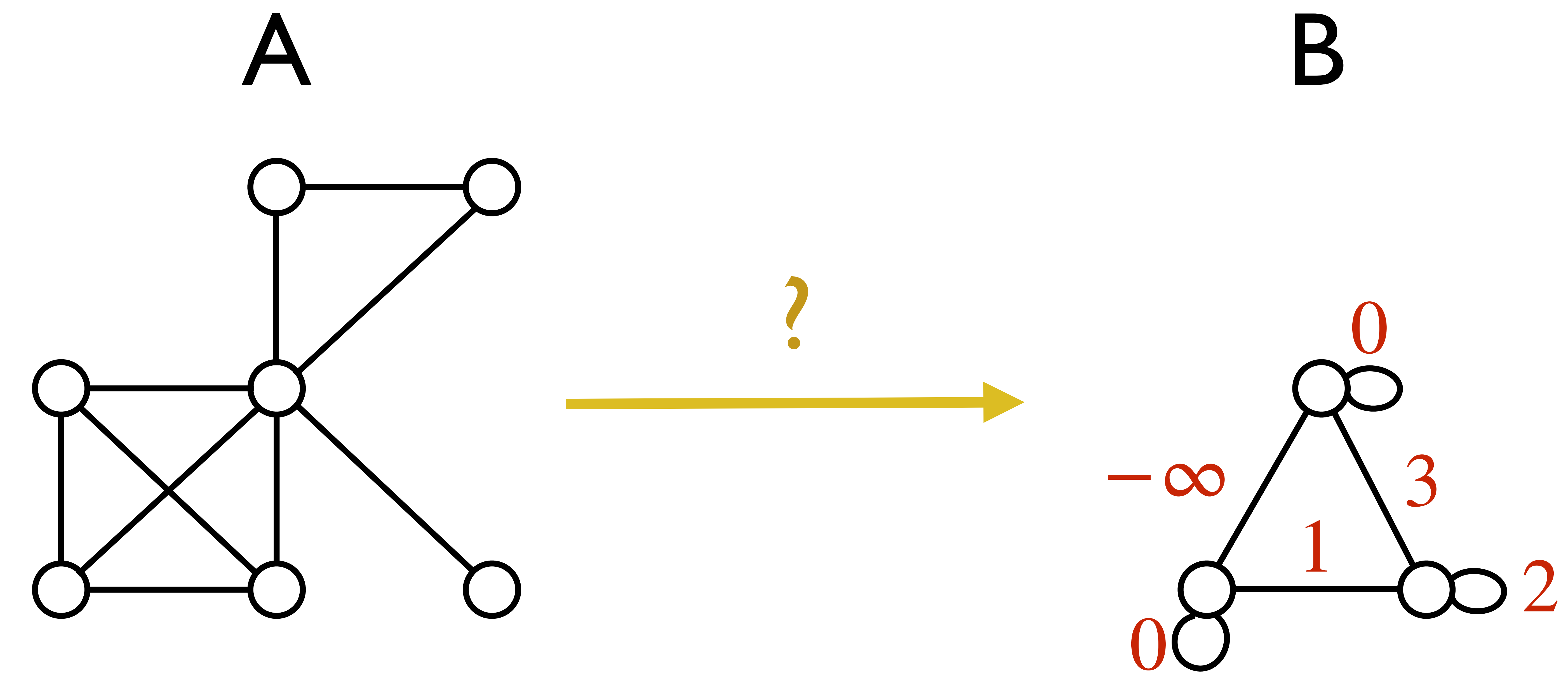

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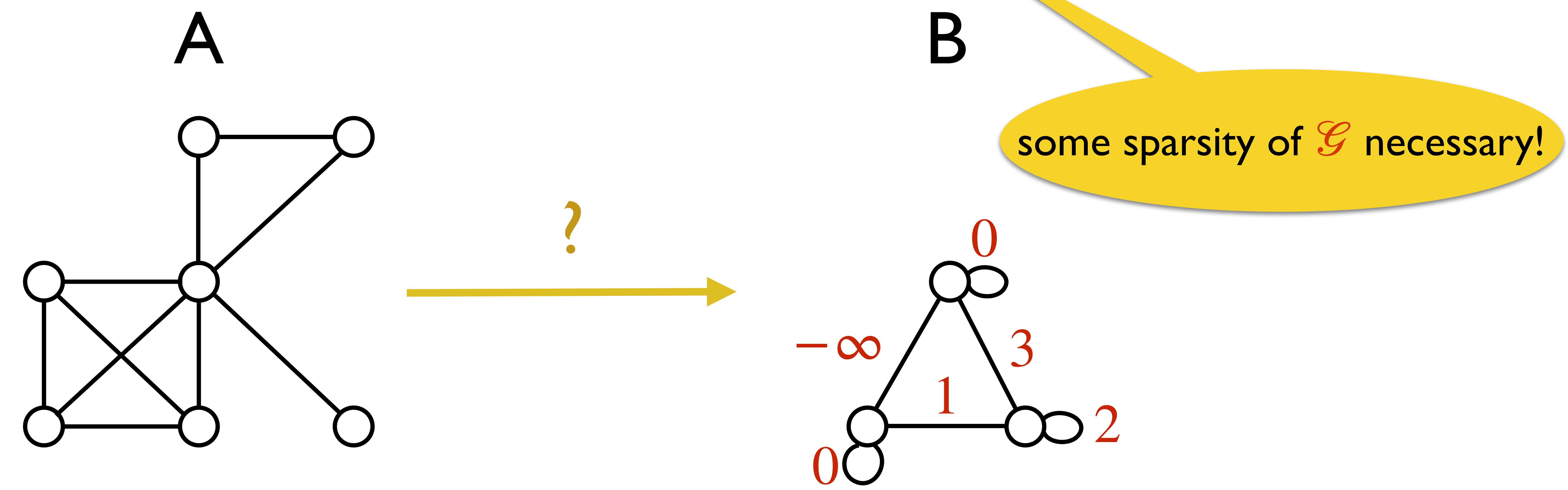

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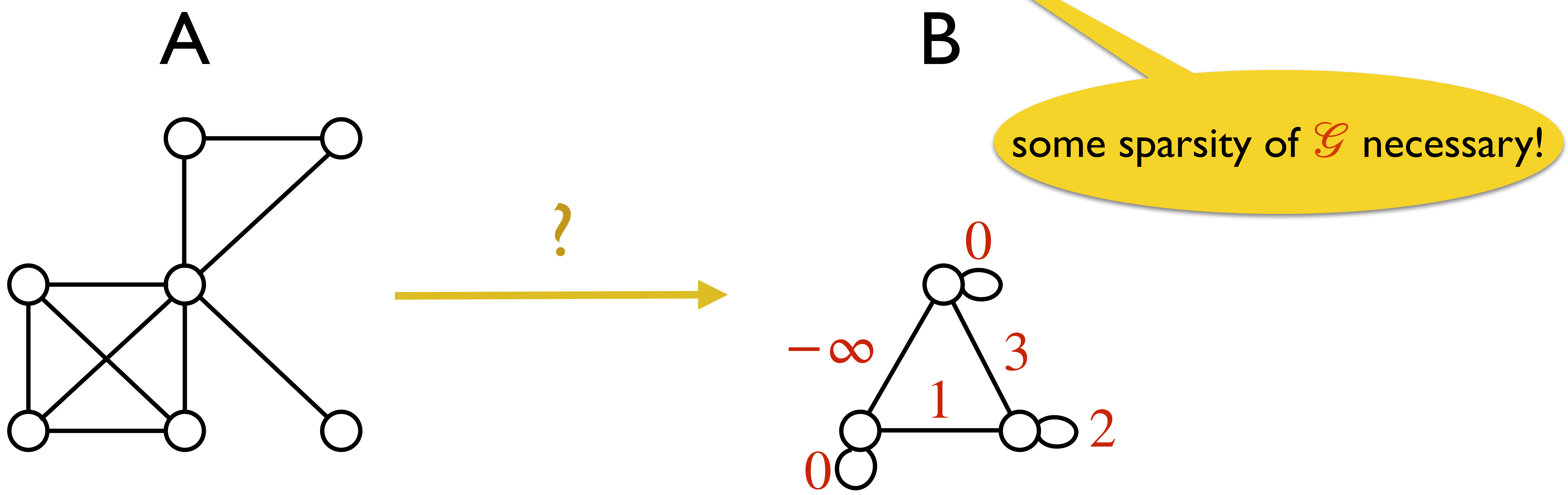

- VCSP(-, B) $\in$ PTIME or NP-complete

[Kozik-Ochremiak ICALP'15 + Kolmogorov et al. SICOMP'17]

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- $\mathscr{G}$ is efficiently Baker *[Dvořák SODA'20]*
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- $\mathbf{x} \leq \mathbf{y}$ and $R(\mathbf{x}) < \infty$ then $R(\mathbf{y}) < \infty$

PTAS for MinVCSP($\mathscr{A}_{\mathscr{G}}$, B)

- $\mathscr{G}$ is efficiently Baker *[Dvořák SODA'20]*
- B is diagonalisable *[Brightwell-Winkler JCTB'00]*

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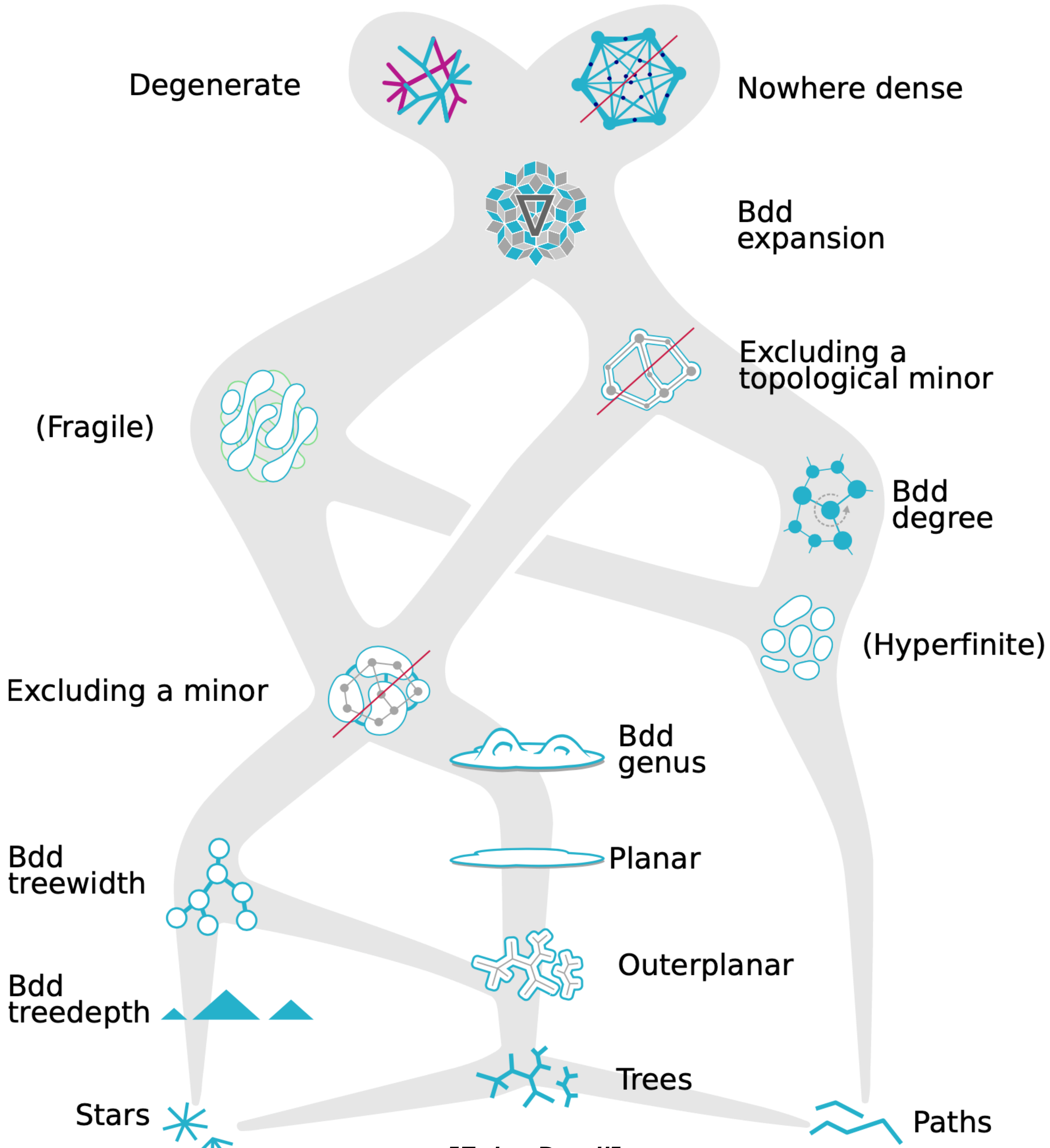
Min-VC on fr-tw-fr?

- Min-VC: domain $\{0,1\}$ with $0 \leq 1$

Conj: MaxCSP($\mathscr{A}_{\mathscr{G}}$,-) admits a PTAS **iff** $\mathscr{G}$ is fr-tw-fr.



[Felix Reidl]

- Gap-ETH-hardness for tournaments

- hardness of non-degenerate $\mathscr{G}$?
- hardness of 3-regular high girth $\mathscr{G}$?
- hardness of $\mathscr{G}$ containing expanders?
- Sherali-Adams gaps?

- O(1)-approx
- weak hyperfiniteness
- EPTAS (via random samples)
- twin-width